Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

SUPERSYMMETRY AND SUPERGRAVITY

Trinity Term 2021

THURSDAY, 22ND APRIL 2021, Opening Time 09:30 am UK Time

You should submit answers to both questions.

You have **2 hours** *writing time to complete the paper and up to* **30 minutes** *technical time for uploading your file. The allotted technical time must not be used to finish writing the paper.*
Mode of completion (format in which you will complete this exam): **handwritten**
You are permitted to use the following material(s):
Calculator (candidate to provide)
The use of computer algebra packages is **not** *allowed.*

1. (a) [8 marks] Consider the $\mathcal{N}$-extended supersymmetry algebra in 4d Minkowski spacetime. Use a 2-component Weyl spinor notation $Q^I{}_\alpha$, $\overline{Q}_{I\dot\alpha}$ for the supercharges.

   - Give the $[J_{\mu\nu}, P_\rho]$ commutator to fix your conventions for the Lorentz generators. Then give the commutators between the supercharges and the Poincaré generators, and the most general form of the anticommutators among supercharges.
   - Verify the $JQ\overline{Q}$ super Jacobi identity by direct computation.

   Hint: the following identity can be useful,

$$\sigma_{\mu\nu}\,\sigma_\rho - \sigma_\rho\,\overline{\sigma}_{\mu\nu} = \eta_{\mu\rho}\,\sigma_\nu - \eta_{\nu\rho}\,\sigma_\mu \ .$$

   (b) [6 marks] Consider minimal 4d $\mathcal{N}=1$ supersymmetry. Prove that any supermultiplet of 1-particle states with definite 4-momentum $p^\mu \neq (0,0,0,0)$ contains an equal number of bosonic and fermionic degrees of freedom. Generalize the argument to $\mathcal{N}$-extended supersymmetry.

   (c) [11 marks] Let us now study 2d $\mathcal{N}=(2,2)$ supersymmetry. The spacetime coordinates are denoted $x^\mu = (x^0, x^1)$. The Minkowski metric is $\eta_{\mu\nu} = \mathrm{diag}(-1,1)$. It is convenient to define $x^\pm = x^0 \pm x^1$, $P_\pm = \frac{1}{2}(P_0 \pm P_1)$. A Lorentz transformation acts as $x^\pm \to e^{\pm\gamma}\,x^\pm$, $P_\pm \to e^{\mp\gamma}\,P_\pm$, where $\gamma \in \mathbb{R}$ is the parameter of the transformation. The supercharges are denoted $Q_\pm$, $\overline{Q}_\pm$. They are 1-component Weyl spinors. Under a Lorentz transformation we have $Q_\pm \to e^{\mp\gamma/2}\,Q_\pm$, $\overline{Q}_\pm \to e^{\mp\gamma/2}\,\overline{Q}_\pm$. We also have $(Q_\pm)^\dagger = \overline{Q}_\pm$. The non-trivial anticommutators in the SUSY algebra are (we do not include central charges)

$$\{Q_+, \overline{Q}_+\} = 2\,P_+ \ , \qquad \{Q_-, \overline{Q}_-\} = 2\,P_- \ . \tag{1}$$

   - Define 2d $\mathcal{N}=(2,2)$ superspace as a coset with coordinates $(x^\pm, \theta^\pm, \overline{\theta}^\pm)$, where $\theta^\pm$, $\overline{\theta}^\pm$ are Grassmann variables with $(\theta^\pm)^* = \overline{\theta}^\pm$. Write the coset representative as

$$G(x^\pm, \theta^\pm, \overline{\theta}^\pm) = \exp(-i x^+ P_+ - i\,x^- P_- + i\,\theta^+ Q_+ + i\,\theta^- Q_- + i\,\overline{\theta}^+\,\overline{Q}_+ + i\,\overline{\theta}^-\,\overline{Q}_-) \ . \tag{2}$$

     How do $\theta^\pm$, $\overline{\theta}^\pm$ transform under the Lorentz transformation $x^\pm \to e^{\pm\gamma}\,x^\pm$?

   - Act with

$$g_0^{-1} := \exp(-i\,\xi^+ Q_+ - i\,\xi^- Q_- - i\,\overline{\xi}^+\,\overline{Q}_+ - i\,\overline{\xi}^-\,\overline{Q}_-) \tag{3}$$

     on $G(x^\pm, \theta^\pm, \overline{\theta}^\pm)$ from the left. Determine the corresponding motion in superspace and the differential operators that generate it. You should find

$$\mathbf{Q}_\pm = i\left(\frac{\partial}{\partial\theta^\pm} + i\,\overline{\theta}^\pm\,\partial_\pm\right) \ , \qquad \overline{\mathbf{Q}}_\pm = i\left(\frac{\partial}{\partial\overline{\theta}^\pm} + i\,\theta^\pm\,\partial_\pm\right) \ . \tag{4}$$

     Hint: the Baker-Campbell-Hausdorff formula is

$$e^A\,e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[A,[A,B]]-\frac{1}{12}[B,[A,B]]\ldots} \ .$$

   - Write down the schematic form of the expansion of a generic, complex superfield $\mathcal{S}(x^\pm, \theta^\pm, \overline{\theta}^\pm)$ in component fields. Assume that the $\theta^\pm = \overline{\theta}^\pm = 0$ component of $\mathcal{S}$ is a Grassmann-even scalar. Count the off-shell degrees of freedom carried by the bosonic and fermionic component fields in $\mathcal{S}$. Determine how the component fields transform under the Lorentz transformation $x^\pm \to e^{\pm\gamma}\,x^\pm$.
   - Consider now the action of g_0^{-1} on $G(x^\pm, \theta^\pm, \overline{\theta}^\pm)$ from the right. Use it to determine the SUSY covariant derivatives $D_\pm$, $\overline{D}_\pm$. Why are they called SUSY covariant derivatives? Normalize them so that they take the form $D_\pm = \frac{\partial}{\partial\theta^\pm} + \ldots$, $\overline{D}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} + \ldots$. Compute the anticommutators among $D_\pm$, $\overline{D}_\pm$.

2. (a) [4 marks] Write down the superspace action for a generic 4d $\mathcal{N}=1$ model (not necessarily renormalizable) that contains only chiral superfields and no vector superfields. (Only consider terms that yield at most two derivatives in spacetime.) Briefly explain the physical content of the terms in the superspace action.

(b) [2 marks] Consider a 4d $\mathcal{N}=1$ SUSY gauge theory in superspace, with a vector superfield V in the adjoint representation of a simple non-Abelian gauge group G and a chiral superfield Φ in the representation $\mathbf{r}$ of the gauge group. Write down the expression for a finite gauge transformation of V and Φ in superspace.

(c) [4 marks] Let us consider a 4d $\mathcal{N}=1$ SUSY gauge theory with gauge group $SU(N_c)$ and flavor symmetry $SU(N_f)$, with chiral multiplets Q, $\widetilde{Q}$, Φ as specified in the following table of representations:

	$SU(N_c)$	$SU(N_f)$
Q	$\square$	$\square$
$\widetilde{Q}$	$\overline{\square}$	$\overline{\square}$
Φ	adj	$\bullet$

(5)

Prove that the 1-loop beta function of the $SU(N_c)$ gauge coupling vanishes if we set $N_f = 2N_c$.

Hint: in suitable units, the gauge fields/ghosts contribute $\frac{11}{3}T(\text{adj})$ *to the 1-loop beta function; a complex scalar in the representation* $\mathbf{r}$ *of the gauge group contributes* $-\frac{1}{3}T(\mathbf{r})$*; a Weyl fermion in the representation* $\mathbf{r}$ *of the gauge group contributes* $-\frac{2}{3}T(\mathbf{r})$*. One also has* $T(\text{adj}) = N$, $T(\square) = T(\overline{\square}) = 1/2$ *for gauge group* $SU(N)$.

(d) [15 marks] Let us consider a model with no gauge fields and two chiral superfields Φ_1 and Φ_2. We assume canonical kinetic terms. We choose the superpotential

$$W = \frac{1}{2}\, M\, \Phi_1^2 + \frac{1}{2}\,\lambda\,\Phi_1\,\Phi_2^2 \ .$$

We assume M, λ are non-zero complex numbers. We use the notation X_i for the $\theta = 0 = \overline{\theta}$ component of the chiral superfield Φ_i.

- Write down the F-term equations and show that the model admits a unique SUSY vacuum. Determine the masses of the quanta of the fields X_1, X_2 around the vacuum, in the tree-level (classical) approximation.
- Study the mass matrix for the fermions in the tree-level (classical) approximation and compare their masses to the scalar masses.
- Let us regard M, λ as background chiral superfields. We want to show that the model has a global internal symmetry $U(1)_A \times U(1)_B \times U(1)_R$, where $U(1)_R$ is an R-symmetry, while $U(1)_{A,B}$ are non-R-symmetries. Complete the following table of charges in such a way that model is indeed invariant under $U(1)_A \times U(1)_B \times U(1)_R$:

	$U(1)_A$	$U(1)_B$	$U(1)_R$
Φ_1	1	0	?
Φ_2	0	1	?
M	?	?	0
λ	?	?	0

(6)

- Let us now study the Wilsonian effective action of the system well below the scale $|M|$. Based on your results on the masses of scalars and bosons, argue that we have to integrate out Φ_1. The effective superpotential W_{eff} is thus going to depend on Φ_2, M, λ. How do SUSY and $U(1)_A \times U(1)_B \times U(1)_R$ symmetry restrict the dependence of W_{eff} on these quantities?

- Study the quantity $\Phi_2^j \, M^k \, \lambda^p$ where j, k, p are real numbers. Argue that there is only one choice for j, k, p such that the term $\Phi_2^j \, M^k \, \lambda^p$ can enter W_{eff}. Fix the numerical coefficient in front of it by matching with the result that one obtains if one integrates out Φ_1 at tree-level level.