A15758W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

QUANTUM MATTER: SUPERCONDUCTORS, SUPERFLUIDS, AND FERMI LIQUIDS

Trinity Term 2021

FRIDAY, 11TH JUNE 2021, Opening Time 09:30 am UK Time

You should submit answers to both questions.

You have **2 hours** *writing time to complete the paper and up to* **30 minutes** *technical time for uploading your file. The allotted technical time must not be used to finish writing the paper.*
Mode of completion (format in which you will complete this exam): **handwritten**
The use of computer algebra packages or a calculator is **not** *allowed.*

1. For a system of mass m bosons with short-range interaction $U\delta(\mathbf{r}-\mathbf{r}')$, with $\delta(\mathbf{r})$ a three-dimensional delta function, the time-dependent Gross-Pitaevskii equation is of the form

$$i\hbar\frac{\partial}{\partial t}\Psi(\mathbf{r},t) = \left[-\frac{\hbar^2}{2m}\nabla^2 - \mu + U|\Psi(\mathbf{r},t)|^2\right]\Psi(\mathbf{r},t) \tag{1}$$

with $\mu = U\bar{n}$ the chemical potential and $\bar{n}$ the condensate density of an unperturbed (stationary) superfluid.

(a) Explain why Eq. 1 can be understood as the time-dependent Hartree approximation. [4]

(b) Give an expression for the mass current density (momentum per volume) in terms of Ψ. Give an expression for the velocity of the fluid in terms of Ψ. [3]

(c) Consider a time-independent solution of Eq. 1 of the form $\Psi = fe^{i\mathbf{k}\cdot\mathbf{r}}$ for some constant scalar f. Show that f drops as velocity increases. What is the maximum velocity that can be obtained with such a solution? What is the maximum current density that can be achieved? (These result may not be in perfect agreement with the Landau criterion). [8]

(d) For superconductors a very similar calculation using the Ginzburg-Landau equations can determine the critical *electrical* current density of thin superconducting wires in fairly good agreement with experiment. Why does the wire need to be thin? How thin must the wire be for such a calculation to be reasonable? What boundary condition must hold? [5]

(e) Any time-independent solution $\Psi_0(\mathbf{r})$ of Eq. 1 can be boosted to any velocity $\mathbf{v}$ to give another solution via

$$\Psi(\mathbf{r},t) = \Psi_0(\mathbf{r}-\mathbf{v}t)\exp\left(\frac{i}{\hbar}m\mathbf{v}\cdot\mathbf{r} - \frac{i}{\hbar}\frac{m|\mathbf{v}|^2 t}{2}\right) \tag{2}$$

(This can be shown by substituting Eq. 2 into Eq. 1. You do not have to do this!) Explain how this Galilean invariance can be reconciled with Landau's argument for a critical velocity in Bose fluids. Under what conditions do you expect Eq. 2 to hold, and under what conditions would you expect a reduction in f with increasing velocity as described in part (c)? [5]

2. Consider the Hamiltonian for spin-$\frac{1}{2}$ fermions in *two dimensions* interacting with each other via an interaction $V(\mathbf{r})$

$$H = \sum_i \frac{\mathbf{p}_i^2}{2m} + \frac{1}{2}\sum_{i \neq j} V(\mathbf{r}_i - \mathbf{r}_j)$$

(a) Consider plane-wave orbitals $\varphi_{\mathbf{k}}(\mathbf{r}) = \frac{1}{\sqrt{\mathcal{A}}} e^{i\mathbf{k}\cdot\mathbf{r}}$ where $\mathcal{A}$ is the area of the system. Let $c^\dagger_{\mathbf{k},\sigma}$ and $c_{\mathbf{k},\sigma}$ be operators that create or annihilate respectively a fermion in the orbital $\varphi_{\mathbf{k}}$ with spin σ. Write the anticommutation relations for these operators. [3]

(b) Rewrite the Hamiltonian in second quantized form in terms of these plane-wave creation and annihilation operators and the Fourier transform of $V(\mathbf{r})$. [5]

For the remainder of this problem we assume an interaction which is the Laplacian of a real-space delta function

$$V(\mathbf{r}) = -V_0 \nabla^2 \delta(\mathbf{r})$$

with $V_0 > 0$ the interaction strength and $\delta(\mathbf{r})$ the two-dimensional delta function.

(c) Consider a zero-temperature situation where the spin-up and spin-down Fermi surfaces are both circular and the densities of spin-up and spin-down fermions are $n_\uparrow$ and $n_\downarrow$ respectively. Calculate the energy of this state in first-order perturbation theory in the interaction as a function of $n_\uparrow$ and $n_\downarrow$. [12]

(d) Assuming a fixed total density $n = n_\uparrow + n_\downarrow$, show that within Hartree-Fock theory, the ground state is either a fully unpolarized system (equal number of spin-up and spin-down fermions) or a fully spin polarized system. At what value of V_0 does the transition between these two states occur? [5]