A15758W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

QUANTUM MATTER: SUPERCONDUCTORS, SUPERFLUIDS, AND FERMI LIQUIDS

Trinity Term 2020

MONDAY, 8TH JUNE 2020, 14:30

You should submit answers to both of the two questions.

You have **3 hours** *to complete the paper and upload your answer file.*
The use of computer algebra packages and calculators is **not** *allowed*

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

1. Consider a Hamiltonian for spinless bosons of mass m in three dimensions interacting via a general interaction $U(\mathbf{r}-\mathbf{r}')$. In lecture and homework we considered $U(\mathbf{r}-\mathbf{r}') = U_0\delta(\mathbf{r}-\mathbf{r}')$. Here we intend to consider arbitrary radially symmetric interaction $U(\mathbf{r}-\mathbf{r}') = U(|\mathbf{r}-\mathbf{r}'|)$. You may assume U is smooth and $\int \mathbf{dr}\, U(\mathbf{r}) > 0$. Only partial credit will be given for considering the special case of the delta function interaction.

   (a) [5 marks] Let $\hat{\psi}^\dagger(\mathbf{r})$ be the operator that creates a boson at position $\mathbf{r}$. Give the commutation relations for the operators $\hat{\psi}$ and $\hat{\psi}^\dagger$, and write down the Hamiltonian of the system in second quantized notation.

   (b) [4 marks] From the Hamiltonian derive the Heisenberg equations of motion for the operator $\hat{\psi}(\mathbf{r})$. Hint: For a time independent operator $\hat{A}$, the Heisenberg equations are $i\hbar d\hat{A}/dt = [\hat{A}, H]$.

   (c) [5 marks] From the Heisenberg equations of motion, and assuming there are many bosons in the system, derive a time-dependent Gross-Pitaevskii (Ginzburg-Landau) equation. Explain your steps and explain the relationship between the *classical* order parameter ψ that occurs in the Gross-Pitaevskii equation and the operator $\hat{\psi}$.

   (d) [8 marks] Write a time-dependent classical order parameter of the form

$$\psi(\mathbf{r}, t) = e^{-i\mu t/\hbar}\left[\psi_0 + a_+ e^{-i\mathbf{k}\cdot r + i\omega t} + a_- e^{i\mathbf{k}\cdot r - i\omega t}\right]$$

   where a_+ and a_- are small (You may take ψ_0 and a_+, and a_- to all be real). Substituting this into your time dependent Gross-Pitaevskii equation derive the dispersion $\omega(\mathbf{k})$.

   (e) [3 marks] From the dispersion you just calculated, what is the sound velocity for this superfluid in terms of the chemical potential and the boson mass? Give a condition involving the form of the interaction U as well as μ and m which would imply that the critical velocity is lower than the sound velocity?

2. Consider a system of electrons hopping among three orbitals arranged in an equilateral triangle with a Hubbard interaction between the electrons which penalizes double occupancy of an orbital. The Hamiltonian takes the form

$$H = -t \sum_{\sigma=\uparrow,\downarrow} \sum_{i \neq j} c^{\dagger}_{i,\sigma} c_{j,\sigma} + U \sum_i \hat{n}_{i,\uparrow} \hat{n}_{j,\downarrow}$$

where

$$\{c_{i,\sigma}\,, c^{\dagger}_{j,\sigma'}\} = \delta_{i,j}\delta_{\sigma,\sigma'}$$

Here $i, j = 0, 1, 2$ are the three different orbitals, $\sigma = \uparrow, \downarrow$ represents the z-component of the electron spin ($\pm\frac{1}{2}$), and $\hat{n}_{i,\sigma} = c^{\dagger}_{i,\sigma} c_{i,\sigma}$ is the number operator that counts electrons on orbital i with spin σ. Assume that t is real and positive.

(a) [3 marks] Expain why both the total number of electrons N and the total z-component of the spin angular momentum S^z_{total} are conserved quantum numbers for this Hamiltonian. Show also that $(S_{total})^2$ is conserved.

For the remainder of this problem consider the system to have fixed particle number $N = 2$.

(b) [7 marks] What is the dimension of the Hilbert space? (Reminder: $N = 2$ electrons). Make a Fourier transform to rewrite the kinetic term of the Hamiltonian in a momentum basis. In the case where $U = 0$ give all of the eigenenergies of the 2 electron system and their corresponding degeneracies. Which of these eigenstates (and how many of them) have their energies unchanged when U is made nonzero?

(c) [7 marks] Make a Fourier transform on the interaction term of the Hamiltonian. Assume U is small. In Hartree-Fock approximation, find the energy of the ground state.

(d) [8 marks] Consider all states in the Hilbert space having total momentum $k = 0$ and spin $S^z_{total} = 0$. Write the Hamiltonian for states within this space as a 3 by 3 matrix. Determine the ground state of this Hamiltonian. (Hint: You should already know one of the eigenvalues and eigenvectors of the 3 by 3 matrix from part (b) above.)