A15088W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

QUANTUM FIELD THEORY

Hilary Term 2021

WEDNESDAY, 13TH JANUARY 2021, Opening Time: 09:30 a.m GMT

You should submit answers to all three questions.

You have **3 hours** *writing time to complete the paper and up to* **1 hour** *technical time for uploading your file. The allotted technical time must not be used to finish writing the paper. The use of a calculator and/or computer algebra packages is* **not** *allowed.*

1. The Lagrangian density for a system consisting of a real scalar and a Dirac fermion of mass m, and a heavier Dirac fermion of mass $M > 2m$, is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{3!}\lambda\phi^3 + \overline{\Psi}(i\gamma^\mu\partial_\mu - M)\Psi + \overline{\chi}(i\gamma^\mu\partial_\mu - m)\chi - g\phi\overline{\Psi}\chi - g\phi\overline{\chi}\Psi.$$

   (a) [4 marks] Write down the momentum-space Feynman rules assuming Minkowski space-time.

   (b) [4 marks] Draw the tree-level Feynman graphs, including momentum labelling, for the following processes: i) scattering of a Ψ fermion and a χ fermion, ii) scattering of a Ψ fermion and a boson, iii) decay of a Ψ fermion into a χ fermion and one boson, iv) decay of a Ψ fermion into a χ fermion and two bosons.

   (c) [6 marks] A Ψ fermion at rest with spin s_1 and four-momentum $p_1 = (M, 0, 0, 0)$ decays into a χ fermion with spin and four-momentum s_2, p_2 and one boson with four-momentum p_3. Write down a formula for the matrix element $M^{s_1,s_2}_{p_1,p_2,p_3}$ corresponding to your diagram in question (b)iii. Explain why the total amplitude squared for the decay of unpolarised particles to a final state of any spin is given by

$$|\mathcal{M}|^2 = \frac{1}{2}\sum_{s_1,s_2} |M^{s_1,s_2}_{p_1,p_2,p_3}|^2.$$

   Show that

$$|\mathcal{M}|^2 = g^2 M(M + 2m).$$

   You may assume that $\sum_s u^s_a(\mathbf{p})\overline{u}^s_b(\mathbf{p}) = (\not{p} + m)_{ab}$ for Dirac spinors $u^s(\mathbf{p})$ and mass m.

   (d) [7 marks] The decay width is given by

$$\Gamma = \frac{1}{2M}\int \frac{d^3\mathbf{p}_1}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}_1}}\int \frac{d^3\mathbf{p}_2}{(2\pi)^3}\frac{1}{2E_{\mathbf{p}_2}}|\mathcal{M}|^2(2\pi)^4\delta^4(p_1 - p_2 - p_3).$$

   Evaluate Γ.

   (e) [4 marks] Now consider the decay of a Ψ fermion into a χ fermion and two bosons. Without detailed calculation, explain how the decay width Γ depends on m in the region $M \simeq 3m$.

2. In the basis

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \tag{1}$$

the Dirac spinors $u^s(\mathbf{p})$ are given by

$$u^s(\mathbf{p}) = \sqrt{E_{\mathbf{p}} + m}\begin{pmatrix} \xi^s \\ \frac{\mathbf{p}.\boldsymbol{\sigma}}{E_{\mathbf{p}}+m}\xi^s \end{pmatrix}, \tag{2}$$

where ξ^s are an orthonormal pair of constant two-component spinors. The Dirac fermion field is given by

$$\psi(t, \mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\sum_s \left(b^{s\dagger}_{-\mathbf{p}}v^s(-\mathbf{p})e^{iE_{\mathbf{p}}t} + a^s_{\mathbf{p}}u^s(\mathbf{p})e^{-iE_{\mathbf{p}}t}\right)e^{i\mathbf{p}.\mathbf{x}}, \tag{3}$$

where the annihilation and creation operators satisfy the standard anti-commutation rules.

   (a) [5 marks] Show by direct calculation that $\sum_s u^s_a(\mathbf{p})\overline{u}^s_b(\mathbf{p}) = (\not{p} + m)_{ab}$. By using the fact that $\{u^s(\mathbf{p}), v^s(-\mathbf{p})\}$ span the spinor space deduce that $\sum_s v^s_a(\mathbf{p})\overline{v}^s_b(\mathbf{p}) = (\not{p} - m)_{ab}$ for Dirac antiparticle spinors $v^s(\mathbf{p})$.

(b) [5 marks] Find the equal time anti-commutator $\{i\overline{\psi}(t,\mathbf{x})\gamma^0, \psi(t,\mathbf{y})\}$. What is the significance of this quantity?

(c) [5 marks] Show that

$$a^s_{\mathbf{p}} = \frac{e^{iE_{\mathbf{p}}t}}{\sqrt{2E_{\mathbf{p}}}} u^s(\mathbf{p})^\dagger \int d^3\mathbf{x}\, e^{-i\mathbf{p}.\mathbf{x}} \psi(t,\mathbf{x}), \tag{4}$$

and write down a similar relationship for $b^{s\dagger}_{-\mathbf{p}}$.

(d) [10 marks] Assuming the Hamiltonian in the form

$$H = \int \frac{d^3\mathbf{p}}{(2\pi)^3} E_{\mathbf{p}} \sum_s (a^{s\dagger}_{\mathbf{p}} a_{\mathbf{p}} - b^s_{\mathbf{p}} b^{s\dagger}_{\mathbf{p}}), \tag{5}$$

use the results of the previous part to find an expression for the Hamiltonian density in position space in terms of the field $\psi(t,\mathbf{x})$.

3. The Lagrangian density for a real scalar field theory in 3 dimensions is given by

$$\mathcal{L} = \frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{g_1}{4!}\phi^4 - \frac{g_2}{6!}\phi^6.$$

(a) [3 marks] What are the dimensions of g_1 and g_2?

(b) [6 marks] By analysing a graph with n external lines, e internal lines, n_4 four-point vertices, and n_6 six-point vertices, show that the superficial degree of divergence is given by

$$\omega = 3 - \frac{1}{2}n - n_4.$$

Explain briefly the significance of this result.

(c) [9 marks] For the two-point function draw i) all one-particle irreducible graphs with one vertex, and ii) all one-particle irreducible graphs with two vertices that contribute to wavefunction renormalization. For each graph write down expressions using momentum-space Feynman rules (there is no need to evaluate the combinatorial factors).

(d) [7 marks] How many different momentum integrals occur? For each integral show how any ultra-violet divergence arises and compare the actual degree of divergence of the graphs with ω.