A15088W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

QUANTUM FIELD THEORY

Hilary Term 2020

WEDNESDAY, 15TH JANUARY, 09:30 am to 12:30 pm

You should submit answers to all three questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. The Lagrangian density for a system consisting of a real scalar and two types of Dirac fermion in four dimensions is given by

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 + \sum_{i=1,2}\overline{\psi}_i(i\gamma^\mu\partial_\mu - m)\psi_i - g\phi\overline{\psi}_i\psi_i$$

   (a) [4 marks] Write down the momentum space Feynman rules assuming Minkowski space-time.

   (b) [6 marks] Draw the tree-level Feynman graphs, including momentum labelling, for the following processes: i) scattering of a type 1 fermion and a type 2 fermion, ii) scattering of two type 1 fermions, iii) scattering of a type 1 fermion and a type 1 anti-fermion.

   (c) [7 marks] A type 1 fermion with spin and four-momenta s_1, p_1 scatters off a type 2 fermion with spin and four-momenta s_2, p_2 producing a type 1 fermion with spin and four-momenta s_3, p_3 and a type 2 fermion with spin and four-momenta s_4, p_4 . Write down a formula for the matrix element $M^{s_1,s_2,s_3,s_4}_{p_1,p_2,p_3,p_4}$ corresponding to your diagram in question (b)i. Explain why the total amplitude squared for the scattering of unpolarised particles to a final state of any spin is given by

$$|M|^2 = \frac{1}{4}\sum_{s_1,s_2,s_3,s_4}|M^{s_1,s_2,s_3,s_4}_{p_1,p_2,p_3,p_4}|^2$$

   Show that

$$|M|^2 = g^4\left(\frac{4m^2 - q^2}{m^2 - q^2}\right)^2$$

   where $q^2 = (p_1 - p_3)^2$. You may assume that $\sum_s u^s_a(p)\overline{u}^s_b(p) = (\not{p} + m)_{ab}$ for Dirac spinors $u^s(p)$.

   (d) [8 marks] Consider the Green's function with n external scalar lines of momenta $p_1, \ldots p_n$ and no external fermions lines, $G^n(p_1, \ldots p_n; g, m)$, where we have displayed explicitly the dependence on the mass and the coupling constant. The naïve scaling dimension d_S is defined by the behaviour of the Green's function when *all* masses and momenta are rescaled by a factor λ through

$$G^n(\lambda p_1, \ldots \lambda p_n; g, \lambda m) = \lambda^{d_S} G^n(p_1, \ldots p_n; g, m).$$

   By analysing a general Feynman graph with n external scalar lines, n_v vertices, n_f internal fermion lines and n_s internal scalar lines, show that $d_S = 4 - n$. Discuss the superficial divergence of graphs and the counter-terms required to renormalize this theory.

2. Consider the complex scalar fields

$$\phi(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\left(b^\dagger_{-\mathbf{p}} + a_\mathbf{p}\right)e^{i\mathbf{p}.\mathbf{x}}, \quad \phi^\dagger(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2E_\mathbf{p}}}\left(a^\dagger_{-\mathbf{p}} + b_\mathbf{p}\right)e^{i\mathbf{p}.\mathbf{x}}$$

   where the annihilation and creation operators satisfy the standard commutation rules. The Hamiltonian and a new operator Q are given by

$$H = \int \frac{d^3\mathbf{p}}{(2\pi)^3} E_\mathbf{p}(a^\dagger_\mathbf{p}a_\mathbf{p} + b^\dagger_\mathbf{p}b_\mathbf{p}), \quad Q = \int \frac{d^3\mathbf{p}}{(2\pi)^3}(a_\mathbf{p} - b_\mathbf{p})(a^\dagger_\mathbf{p} - b^\dagger_\mathbf{p}).$$

   (a) [5 marks] Show that

$$[Q, a_\mathbf{q} + sb_\mathbf{q}] = -(1-s)(a_\mathbf{q} - b_\mathbf{q}), \qquad [Q, a^\dagger_\mathbf{q} + sb^\dagger_\mathbf{q}] = (1-s)(a^\dagger_\mathbf{q} - b^\dagger_\mathbf{q})$$

   where s is a real constant.

(b) [5 marks] Define $\mathcal{C}_\lambda = e^{i\lambda Q}$ where λ is real. For operators A such that $[Q, A] = cA$ where c is a constant show that $A(\lambda) = \mathcal{C}_\lambda A \mathcal{C}_\lambda^\dagger$ satisfies

$$\frac{\partial A(\lambda)}{\partial \lambda} = icA(\lambda).$$

Solve this differential equation to find $A(\lambda)$ in terms of λ and A.

(c) [7 marks] Using the result of parts a) and b) show that

$$\mathcal{C}_{\frac{\pi}{2}}\, \phi(\mathbf{x})\, \mathcal{C}_{\frac{\pi}{2}}^\dagger = \phi^\dagger(\mathbf{x})$$

and deduce that therefore $\mathcal{C}_{\frac{\pi}{2}}$ can be identified as the charge conjugation operator. What is $\mathcal{C}_{\frac{\pi}{2}}\, H\, \mathcal{C}_{\frac{\pi}{2}}^\dagger$?

(d) [8 marks] The parity operator $\mathcal{P}$ has the property that

$$\mathcal{P}\phi(\mathbf{x})\mathcal{P}^\dagger = \phi(-\mathbf{x}).$$

What is the action of $\mathcal{P}$ on annihilation and creation operators? For a single real scalar field find a representation for $\mathcal{P}$ in terms of annihilation and creation operators.

3. The path integral for the generating functional of a real scalar field theory in 4 dimensions is given by

$$Z[\lambda; j] = \int D\phi \exp\left(i \int d^4x \left[\frac{1}{2}\partial^\mu\phi\partial_\mu\phi - \frac{1}{2}(m^2 - i\epsilon)\phi^2 - \frac{\lambda}{3!}\phi^3 + j\phi \right] \right)$$

(a) [10 marks] By formally expanding the relevant exponentials, differentiating and resumming, show that

$$Z[\lambda; j] = \exp\left(\int d^4y\, \frac{\lambda}{3!} \frac{\delta^3}{\delta j(y)^3} \right) Z[0; j].$$

Show that

$$Z[0; j] = Z[0; 0] \exp\left(-\frac{i}{2} \int d^4x \int d^4y \int \frac{d^4p}{(2\pi)^4} \frac{j(x)j(y)e^{-ip\cdot(x-y)}}{p^2 - m^2 + i\epsilon} \right).$$

What is the role of the $i\epsilon$ term in this quantity?

(b) [7 marks] Write down an expression for the expectation value of $T\prod_{i=i}^m \phi(x_i)$ in terms of $Z[\lambda; j]$ and explain how combinatorial factors arise in the Feynman graph expansion.

(c) [8 marks] Compute the combinatorial factors for the following graphs: