A15088W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

QUANTUM FIELD THEORY

Hilary Term 2019

WEDNESDAY, 9TH JANUARY 2019, 09:30 am to 12:30 pm

You should submit answers to all three questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. An experiment finds a quasiparticle for which the relation between energy and momentum is

$$E = \sqrt{\kappa^2|\vec{p}|^4 + m^2} \ ,$$

where κ and m are fixed parameters. Note the quartic power of $|\vec{p}|$, indicating that this theory is not Lorentz invariant.

(a) (8 marks) Find an action for a local field theory that, when quantized, will describe free bosonic quasiparticles with this dispersion relation. As an intermediate step, you should write down the classical Hamiltonian.

(b) (9 marks) Derive the classical equation of motion of your action (note that due to the unconventional form of the action, you probably want to go back and rederive the appropriate Euler-Lagrange equation). Find the plane-wave solution and check that it has the appropriate dispersion relation.

(c) (8 marks) Write an expression for the quantized field in terms of momentum-space creation and annhilation operators satisfying the standard commutation relation for such operators.

2. In this problem you will analyse an interacting field theory involving two real scalar fields Φ and ϕ with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\Big((\partial_\mu\phi)(\partial^\mu\phi) + (\partial_\mu\Phi)(\partial^\mu\Phi) - m^2\phi^2 - M^2\Phi^2 + \lambda\phi\Phi^2 + g\Phi(\partial_\mu\phi)(\partial^\mu\phi)\Big) \ ,$$

where g and λ are couplings, and m and M are masses. Note the derivatives in the interactions in the last term.

(a) (15 marks) Write down the momentum-space Feynman rules, including the propagators for this theory. Make sure you label precisely which (if any) momenta appear.

(b) (10 marks) Compute the scattering amplitude for $\phi\Phi \to \phi\Phi$ to leading non-trivial order in the couplings λ and g (assume the two are the same order).

3. The purpose of this problem is to define and study the Dirac field theory in one spatial dimension, with action

$$S = \int d^2x\, \overline{\psi}(i\partial\!\!\!/ - m)\psi \ , \tag{1}$$

where $\overline{\psi} \equiv \psi^\dagger\gamma^0$ and $\partial\!\!\!/ \equiv \gamma^\mu\partial_\mu$. There are two gamma matrices γ^0 and γ^1 that obey the usual algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$$

where $\eta^{\mu\nu}$ is the usual Minkowski metric restricted to 1+1 dimensions. The "chirality" operator is defined as

$$\gamma^5 \equiv \gamma^0\gamma^1 \ .$$

It is named γ^5 in analogy with the 3+1d case.

(a) (5 marks) Show that γ^5 anticommutes with the other two gamma matrices, and squares to 1. Give an explicit representation of this algebra in terms of 2×2 matrices such that γ^5 is diagonal. Find the eigenvalues of γ^5 in this basis. Use this basis for the remainder of the problem.

(b) (5 marks) Using these gamma matrices, write down the Dirac equation for a two-component spinor ψ in 1+1 dimensions. Check that it indeed yields solutions with energy-momentum relation $E^2 = p^2 + m^2$. (Note here p is not shorthand for a 4-vector – momentum really is just a one-dimensional vector, i.e. a number.)

(c) (5 marks) Let ψ_R and ψ_L be the components of ψ with eigenvalues $+1$ and -1 respectively under γ^5. Write out the Dirac equation in terms of ψ_L and ψ_R, and show that in the massless case $m = 0$, it becomes two independent equations for the ψ_R and ψ_L.

(d) (5 marks) From examining these equations, show that ψ_R corresponds to "right-movers" and ψ_L "left-movers". For example, if at $t = 0$, $\psi_R(0, x) = f(x)$ for some wave packet $f(x)$, the wave packet simply moves to the right at the speed of light.

(e) (5 marks) Finally, consider Lorentz transformations in $1 + 1d$. Since there are no rotations, there are only boosts. Show that a boost of velocity v acts on 2-vectors as

$$\begin{pmatrix} t \\ x \end{pmatrix} \to \frac{1}{\sqrt{1-v^2}} \begin{pmatrix} 1 & v \\ v & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix} .$$

Show that this boost matrix can be rewritten as

$$\begin{pmatrix} \cosh(\alpha) & \sinh(\alpha) \\ \sinh(\alpha) & \cosh(\alpha) \end{pmatrix}$$

for real α. Find the functions $\Lambda_L(\alpha)$ and $\Lambda_R(\alpha)$ such that the action (1) is invariant under boosts that transform the spinor components as

$$\psi_L \to \Lambda_L(\alpha)\psi_L , \qquad \psi_R \to \Lambda_R(\alpha)\psi_R .$$