A15088W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

QUANTUM FIELD THEORY

Hilary Term 2018

WEDNESDAY, 10TH JANUARY 2018, 09:30am to 12:30pm

You should submit answers to three out of the four questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. Consider a free field theory in Minkowski space with real scalar fields ϕ_k, $k = 1, 2$, and Lagrangian density

$$\mathcal{L}_0 = \frac{1}{2}\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 - \frac{1}{2}m^2(\phi_1^2 + \phi_2^2)\,.$$

   (a) [7 marks] Show that this Lagrangian has a symmetry generated by the infinitesimal variations

$$\phi_1 \to \phi_1 + \alpha\,\phi_2\,, \qquad \phi_2 \to \phi_2 - \alpha\,\phi_1\,.$$

   Using Noether's theorem find the conserved charge Q associated with this symmetry.

   [The Noether current associated to a continuous symmetry is given by

$$J^\mu = \sum_k \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi_k)}\delta\phi_k - \tilde{J}^\mu\,, \qquad \textit{with } \delta\mathcal{L} = \partial_\mu\tilde{J}^\mu\,.$$

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   (b) [10 marks] Assuming now that the field theory is canonically quantized, express Q in terms of the fields ϕ_k and the conjugate momenta π_k. Using the expansions

$$\begin{aligned}\phi_k(\vec{\mathbf{x}}) &= \int \tfrac{d^3\vec{\mathbf{p}}}{(2\pi)^3}\tfrac{1}{\sqrt{2E_{\vec{\mathbf{p}}}}}\left(a_{k,\vec{\mathbf{p}}} + a^\dagger_{k,-\vec{\mathbf{p}}}\right)e^{i\vec{\mathbf{p}}\cdot\vec{\mathbf{x}}}\,,\\ \pi_k(\vec{\mathbf{x}}) &= -i\int \tfrac{d^3\vec{\mathbf{p}}}{(2\pi)^3}\sqrt{\tfrac{E_{\vec{\mathbf{p}}}}{2}}\left(a_{k,\vec{\mathbf{p}}} - a^\dagger_{k,-\vec{\mathbf{p}}}\right)e^{i\vec{\mathbf{p}}\cdot\vec{\mathbf{x}}}\,,\end{aligned}$$

   show that

$$Q = -i\int\frac{d^3\vec{\mathbf{p}}}{(2\pi)^3}\left(a^\dagger_{2,\vec{\mathbf{p}}}a_{1,\vec{\mathbf{p}}} - a^\dagger_{1,\vec{\mathbf{p}}}a_{2,\vec{\mathbf{p}}}\right)\,.$$

   Find one-particle states that are eigenstates of Q, and determine their eigenvalues.

   (c) [8 marks] Suppose that the interaction term

$$\mathcal{L}_{\text{int}} = -\lambda\,\phi_1^4 - 2\kappa\,\phi_1^2\,\phi_2^2 - \lambda\,\phi_2^4\,,$$

   is added to $\mathcal{L}_0$. Which inequalities must λ and κ satisfy for the interacting theory to have a stable vacuum (for the purpose of this exercise you can assume that the potential does not acquire quantum corrections)? For which values of λ and κ is the charge Q still conserved in the interacting theory?

2. A real scalar field theory in D-dimensional Euclidean space has Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_0\partial^\mu\phi_0 + \frac{1}{2}m_0^2\phi_0^2 + \frac{\lambda_{2k}}{(2k)!}\phi_0^{2k}\,,$$

   where $k > 1$ is a positive integer.

   (a) [4 marks] Assuming that the action is a dimensionless quantity, find the mass dimension of the coupling λ_{2k}. By introducing a regularization mass scale μ define a dimensionless coupling constant g_{2k}.

   (b) [4 marks] Explain (without derivation) a set of rules for calculating the N-point Euclidean momentum space Green's function $\tilde{G}^{(N)}(p_1, \ldots, p_N)$. Your answer should explain which types of diagrams one needs to consider and how to associate an algebraic expression to each diagram.

(c) [7 marks] The n-point vertex function $\Gamma^{(N)}(p_1, \ldots, p_N)$ is defined as

$$\tilde{G}^{(N)}(p_1, \ldots, p_N) = \prod_{j=1}^{N} \left(\frac{1}{p_j^2 + m^2} \right) \times \Gamma^{(N)}(p_1, \ldots, p_N) \, .$$

For an arbitrary integer $k > 1$, draw the connected Feynman diagram which contributes to the two-point function $\Gamma^{(2)}(p_1, p_2)$ at linear order in g_{2k} and evaluate it using dimensional regularization. Find the leading divergence in four dimensions as a pole in $\epsilon = \frac{4-D}{2}$.
[You may make use of the Gamma function expansion

$$\Gamma(-n + x) = \frac{(-1)^n}{n!} \left(\frac{1}{x} + 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \gamma \right) + \mathcal{O}(x) \, ,$$

and of the following integral:

$$\int d^D p \, e^{-\alpha p^2} = \left(\frac{\pi}{\alpha} \right)^{\frac{D}{2}} \, .$$

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(d) [6 marks] Consider the following change of variables

$$\phi_0 = \sqrt{Z} \phi \, , \qquad Z m_0^2 = m^2 + \delta m^2 \, , \qquad Z^k \lambda_{2k} = \tilde{\lambda}_{2k} + \delta \lambda_{2k} \, ,$$

where $Z = 1 + \delta Z$. Implement this in the Lagrangian and write down the explicit form of the counterterm contribution to the propagator. Setting $k = 2$ and using the result from part c) write down the one-loop contribution to δZ and δm^2 in the minimal subtraction scheme.

(e) [4 marks] Set $D = 4 - 2\epsilon$ and $k = 2$. Assuming that the two-loop contribution to the field renormalization is

$$\delta Z^{(2)} = \frac{g_4^2}{6(32\pi^2)^2} \frac{1}{\epsilon} \, ,$$

find the anomalous dimension $\gamma(g_4)$ for the field ϕ up to order g_4^2 using the formula

$$\gamma(g_4) = -\mu \frac{\partial}{\partial \mu} \log Z \big|_{\lambda_4} \, .$$

3. Consider a quantum field theory in D space-time dimensions.

(a) [6 marks] By exhibiting the relevant Feynman diagrams, but without performing a detailed calculation, explain why in ϕ^4 theory the bare coupling λ_0 is related to a renormalized dimensionless coupling g by the formula

$$\lambda_0 = (\mu^2)^{2 - \frac{D}{2}} \left(g + g^2 \frac{C}{D - 4} + \ldots \right) \, ,$$

where μ is a mass scale parameter and C is a constant which you are not required to calculate.

(b) [6 marks] You may assume that

$$C = -\frac{3}{16\pi^2} \, .$$

Show that if μ is varied but the theory remains unchanged then the renormalised coupling must change in the following way

$$\mu \frac{dg}{d\mu} \big|_{\lambda_0} = \beta(g) \, ,$$

where, to order $\mathcal{O}(g^2)$,

$$\beta(g) = -(4 - D) g + \frac{3}{16\pi^2} g^2 \, .$$

(c) [6 marks] Using the explicit form of the β-function show that in four dimensions

$$g(\mu) = g_s \left(1 - \frac{3g_s}{16\pi^2} \log \frac{\mu}{\mu_s}\right)^{-1},$$

where g_s is the value of the renormalised coupling when $\mu = \mu_s$.

(d) [7 marks] Consider a situation where a more detailed knowledge of the function $\beta(g)$ reveals that it has a zero at $g = g^*$. Explain the ideas of ultraviolet and infrared stability in relation to the fixed point g^*. Show that when $D < 4$ the ϕ^4 theory at order $\mathcal{O}(g^2)$ has an infrared stable fixed point at

$$g^* = (4 - D)\frac{16\pi^2}{3}.$$

4. Consider the following action for an interacting complex scalar field in D space-time dimensions

$$S = \int d^D x \left(\frac{1}{2}\partial_\mu \phi \partial^\mu \phi^* - \frac{1}{2}m^2 \phi\phi^* - \frac{1}{4}\lambda(\phi\phi^*)^2\right).$$

(a) [8 marks] For free theory, i.e. $\lambda = 0$, the field can be canonically quantized and expanded in the Heisenberg picture as

$$\phi(x) = \int \frac{d^3\vec{\mathbf{p}}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{\mathbf{p}}}}} \left(a_{\vec{\mathbf{p}}} e^{ip\cdot x} + b^\dagger_{\vec{\mathbf{p}}} e^{-ip\cdot x}\right)\Big|_{p_0 = E_{\vec{\mathbf{p}}}},$$

$$\phi^*(x) = \int \frac{d^3\vec{\mathbf{p}}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{\mathbf{p}}}}} \left(a^\dagger_{\vec{\mathbf{p}}} e^{-ip\cdot x} + b_{\vec{\mathbf{p}}} e^{ip\cdot x}\right)\Big|_{p_0 = E_{\vec{\mathbf{p}}}},$$

in terms of two sets of harmonic oscillators with commutation relations

$$[a_{\vec{\mathbf{p}}}, a^\dagger_{\vec{\mathbf{q}}}] = [b_{\vec{\mathbf{p}}}, b^\dagger_{\vec{\mathbf{q}}}] = (2\pi)^3\delta^3(\vec{\mathbf{p}} - \vec{\mathbf{q}}), \quad [a_{\vec{\mathbf{p}}}, a_{\vec{\mathbf{q}}}] = [a^\dagger_{\vec{\mathbf{p}}}, a^\dagger_{\vec{\mathbf{q}}}] = [b_{\vec{\mathbf{p}}}, b_{\vec{\mathbf{q}}}] = [b^\dagger_{\vec{\mathbf{p}}}, b^\dagger_{\vec{\mathbf{q}}}] = 0.$$

Consider all possible propagators

$$\langle 0|T\{\phi(x)\phi(y)\}|0\rangle, \quad \langle 0|T\{\phi(x)\phi^*(y)\}|0\rangle, \quad \langle 0|T\{\phi^*(x)\phi(y)\}|0\rangle, \quad \langle 0|T\{\phi^*(x)\phi^*(y)\}|0\rangle,$$

where $T\{\phi(x)\phi(y)\}$ denotes time ordering, and evaluate them in free theory. Express your results in a manifestly Lorentz invariant form and relate them to the Feynman propagator.

(b) [9 marks] The N-point correlation functions in interacting theory are defined using path integrals as

$$\langle \Phi(y_1)\ldots\Phi(y_N)\rangle = \frac{\int \mathcal{D}\phi\mathcal{D}\phi^*\, \Phi(y_1)\ldots\Phi(y_N)\, e^{iS}}{\int \mathcal{D}\phi\mathcal{D}\phi^*\, e^{iS}}, \qquad (\star)$$

where $\Phi(y_i)$ can be either the field $\phi(y_i)$ or its complex conjugate $\phi^*(y_i)$.

Decompose the action into the free-field and the interacting part and write e^{iS} as a power expansion in λ. Write the numerator and the denominator of the N-point correlator $(\star)$ as a sum of free-theory correlators. Using Wick's theorem and the explicit form of free-theory two-point correlation functions you found in part a) to evaluate the numerator and the denominator of $\langle\phi(y_1)\phi^*(y_2)\rangle$ up to order λ. Do not attempt to perform space-time integrals. Give a diagrammatic representation of various terms in the expansion. Show that $\langle\phi(y_1)\phi^*(y_2)\rangle$ receives contributions only from connected diagrams.

(c) [8 marks] Consider all four-point correlation functions $\langle\Phi(y_1)\Phi(y_2)\Phi(y_3)\Phi(y_4)\rangle$ in this theory and argue which of them evaluate to zero. Choose a four-point correlation function which does not evaluate to zero and determine all connected graphs, complete with their appropriate symmetry factors, which contribute to this correlation function up to order λ^2. Write down your answer in terms of D-dimensional space-time integrals. Do not attempt to perform these integrals.