A15088W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

QUANTUM FIELD THEORY

Hilary Term 2017

WEDNESDAY, 11th JANUARY 2017, 9:30am to 12:30pm

You should submit answers to three out of the four questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. Consider a theory for a single complex scalar field ϕ in Minkowski space, with Lagrangian density $\mathcal{L}$ given by

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - f(\phi^* \phi), \tag{1}$$

where f can be any function.

(a) [4 marks] Find equations of motion for ϕ and ϕ^*.

(b) [6 marks] Show that this Lagrangian is invariant under

$$\phi \to \phi' = e^{i\alpha}\phi, \qquad \phi^* \to \phi'^* = e^{-i\alpha}\phi^*, \tag{2}$$

where α is an arbitrary real constant. Show that the Noether current J^μ associated with this symmetry is independent of the function f and may be written as

$$J^\mu = i(\partial^\mu \phi^*)\phi - i\phi^*(\partial^\mu \phi). \tag{3}$$

Show that it is conserved, namely $\partial_\mu J^\mu = 0$.

[The Noether current associated to a continuous symmetry is given by

$$J^\mu = \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi)}\delta\phi + \frac{\delta \mathcal{L}}{\delta(\partial_\mu \phi^*)}\delta\phi^* - \tilde{J}^\mu, \qquad \textit{with } \delta\mathcal{L} = \partial_\mu \tilde{J}^\mu. \tag{4}$$

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(c) [9 marks] Consider the following expansion of field ϕ in the quantum theory:

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\vec{p}}}}(b_{\vec{p}}\, e^{-i\,p\,x} + c^\dagger_{\vec{p}}\, e^{i\,p\,x})\Big|_{p_0=\omega_{\vec{p}}} \tag{5}$$

with

$$[b_{\vec{p}}, b^\dagger_{\vec{q}}] = [c_{\vec{p}}, c^\dagger_{\vec{q}}] = (2\pi)^3 \delta^{(3)}(\vec{p} - \vec{q}) \tag{6}$$

and all other commutators of $b_{\vec{p}}$, $b^\dagger_{\vec{p}}$, $c_{\vec{p}}$ and $c^\dagger_{\vec{p}}$ are zero. By replacing the classical field by the quantum field operator (5) find an expression for the conserved charge operator $Q = \int d^3\vec{x}\, J^0$ in terms of the annihilation and creation operators. Give an interpretation of the charge Q.

(d) [6 marks] Show that $[Q, \phi]$ is proportional to ϕ and find the constant of proportionality. Give an interpretation of this commutation relation.

2. Consider the theory given by the action:

$$\mathcal{S} = \int d^d x \left[\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) - \frac{1}{2}m^2\phi^2 - \frac{\kappa}{3!}\phi^3 - \frac{\lambda}{4!}\phi^4\right], \tag{7}$$

where ϕ is a real scalar field in the Minkowski space.

(a) [4 marks] Assuming that the action is a dimensionless quantity, find the mass dimension of the field ϕ and the coupling constants λ and κ. Is there a critical dimension in which both couplings are dimensionless?

(b) [8 marks] Using Wick's theorem express contributions coming from connected graphs to

$$\langle \phi(x)\phi(y) \rangle$$

up to one-loop order. Do not attempt to perform space time integrals. Give a diagrammatic representation of various terms in the expansion.

(c) [7 marks] Determine all one-particle irreducible one-loop graphs, complete with their appropriate symmetry factors, which contribute to the three-point correlation function in position space

$$\langle \phi(x)\phi(y)\phi(z)\rangle$$

expressing your answer in terms of integrals over d-dimensional momenta. Do not attempt to evaluate the integrals.

(d) [6 marks] Determine all one-loop graphs, complete with their appropriate symmetry factors, which contribute to the four-point vertex function $\Gamma^{(4)}(p_1, p_2, p_3, p_4)$ with $\sum_i p_i = 0$. Write down your answer in terms of integrals over d-dimensional loop momenta. Do not attempt to evaluate the integrals.

3. Consider the theory describing an interacting complex massive scalar field in the Minkowski space, with Lagrangian density

$$\mathcal{L} = (\partial_\mu \phi^*)(\partial^\mu \phi) - m_0^2\, \phi^* \phi - \frac{\lambda_0}{4}(\phi^* \phi)^2 \,. \tag{8}$$

(a) [4 marks] Write down the momentum-space Feynman rules for the propagators and the vertex of the theory. Remember that a simple line for the propagator is not sufficient.

(b) [4 marks] Using $\phi = (\phi_1 + i\phi_2)/\sqrt{2}$, rewrite the Lagrangian in terms of two real scalar fields.

(c) [4 marks] Write down the Feynman rules for the Lagrangian written in terms of ϕ_1 and ϕ_2.

(d) [4 marks] Let us now study the renormalization (at one loop) of this theory, in the picture where there is one complex scalar field. Consider the theory in $d = 4 - \epsilon$ dimensions. The renormalized mass m and renormalized dimensionless coupling g are related to the bare ones in the following way:

$$\lambda_0 = g\mu^\epsilon (1 + \delta_g), \qquad m_0^2 = m^2 + \delta_{m^2} \,. \tag{9}$$

Draw the diagrams for the tree-level and one-loop expansion for the two-point and four-point vertex functions, $\Gamma^{(2)}$ and $\Gamma^{(4)}$, including the counterterms. What is the order in g of the leading one-loop contributions to δ_g and δ_{m^2}?

(e) [5 marks] Compute the divergent part coming from 1-loop diagram(s) of $\Gamma^{(2)}$ in dimensional regularization, and hence show that the mass counterterm in the minimal subtraction scheme is

$$\delta_{m^2} = \frac{g m^2}{8\pi^2 \epsilon} \,. \tag{10}$$

[You may make use of the Gamma function expansion and of the following symmetric Lorentzian integral:

$$\int \frac{d^d q}{(2\pi)^d} \frac{1}{[q^2 + 2q\cdot p - m^2 + i\epsilon']^\alpha} = (-1)^\alpha \frac{i}{(4\pi)^{d/2}} \frac{\Gamma(\alpha - \frac{1}{2})}{\Gamma(\alpha)} (p^2 + m^2)^{d/2 - \alpha} \tag{11}$$

$$\Gamma(-n + x) = \frac{(-1)^n}{n!}\left(\frac{1}{x} + 1 + \frac{1}{2} + \ldots + \frac{1}{n} - \gamma\right) + \mathcal{O}(x) \,. \tag{12}$$

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(f) [4 marks] Find an expression for the renormalized mass m^2 in terms of the bare quantities m_0^2 and λ_0 (at leading order in g). Compute the γ_m function of the theory, defined as

$$\gamma_m(g) = \frac{1}{2}\mu \frac{\partial \log m^2}{\partial \mu}\Big|_{\lambda_0} \,. \tag{13}$$

4. For the quantum field theory of a single massless field ϕ and a single dimensionless coupling constant g, at a regularization scale μ, consider the n-point vertex function $\Gamma^{(n)}(\{p_k\}, g(\mu), \mu)$. Let $\Gamma_0^{(n)}(\{p_k\}, \lambda_0)$ be the bare vertex function depending on the bare coupling λ_0 and on the regulator ϵ. The renormalized vertex function can be written in terms of the bare one as

$$\Gamma^{(n)}(\{p_k\}, g(\lambda_0, \mu), \mu) = Z_\phi(\lambda_0, \mu)^{n/2}\, \Gamma_0^{(n)}(\{p_k\}, \lambda_0)\,, \tag{14}$$

where $\Gamma^{(n)}$ is finite when the regulator ϵ is removed, namely, all divergences are absorbed into the definition of the renormalized coupling $g(\lambda_0, \mu)$ and of the field renormalization $Z_\phi(\lambda_0, \mu)$.

(a) [5 marks] Derive the Callan-Symanzik equation for a generic n-point vertex function $\Gamma^{(n)}$:

$$\left(\mu\frac{\partial}{\partial\mu} + \beta(g)\frac{\partial}{\partial g} - \frac{n}{2}\gamma(g)\right)\Gamma^{(n)}(\{p_k\}, g, \mu) = 0\,, \tag{15}$$

where

$$\beta(g) = \mu\frac{\partial}{\partial\mu}g(\mu)\Big|_{\lambda_0}, \qquad \gamma(g) = \mu\frac{\partial \log Z_\phi}{\partial\mu}\Big|_{\lambda_0}. \tag{16}$$

(b) [5 marks] Assuming that the Callan-Symanzik equation holds for the bare vertex function $\Gamma_0^{(n)}$, namely for $\hat{\Gamma}^{(n)}(\{p_k\}, g_0, \mu) = \Gamma_0^{(n)}(\{p_k\}, \lambda_0)$ with $g_0 = \lambda_0\mu^{-\epsilon}$, show that

$$\beta_0(g_0) = -\epsilon\, g_0, \qquad \gamma_0(g_0) = 0\,. \tag{17}$$

[You may use the definition of $\beta(g)$ directly to calculate $\beta_0(g_0)$.]

(c) [5 marks] Assume that $g(\lambda_0, \mu) = g(g_0)$ and $Z_\phi(\lambda_0, \mu) = Z_\phi(g_0)$ and use

$$\hat{\Gamma}^{(n)}(\{p_k\}, g_0, \mu) = Z_\phi(\lambda_0, \mu)^{-n/2}\, \Gamma^{(n)}(\{p_k\}, g(\lambda_0, \mu), \mu) \tag{18}$$

to show that the Callan-Symanzik equation holds also for $\Gamma^{(n)}$ with coefficients

$$\beta(g(g_0)) = \beta_0(g_0)\frac{\partial g}{\partial g_0}, \qquad \gamma(g(g_0)) = \gamma_0(g_0) + \frac{\beta_0(g_0)}{Z_\phi(g_0)}\frac{\partial Z_\phi}{\partial g_0}\,. \tag{19}$$

(d) [5 marks] Consider the perturbative expansion of the coupling constant and of the field renormalization of the form

$$g(g_0) = g_0 + g_0^2\left(\frac{a_1}{\epsilon} + a_2 + a_3\epsilon + \dots\right), \tag{20}$$

$$Z_\phi(g_0) = 1 + g_0\left(\frac{z_1}{\epsilon} + z_2 + z_3\epsilon + \dots\right). \tag{21}$$

Compute $\beta(g)$ and $\gamma(g)$ and check that they are finite in the absence of regulator.

(e) [5 marks] Now consider the next perturbative order for the coupling constant and focus on the leading order in ϵ, that is

$$g(g_0) = g_0 + g_0^2\frac{a_1}{\epsilon} + g_0^3\frac{b_1}{\epsilon^2}\,. \tag{22}$$

Compute $\beta(g)$ to order g^3 and $\frac{1}{\epsilon}$ and show that its finiteness implies that the coefficients entering the two loop corrections are dependent on the lower order ones.