A15088W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

QUANTUM FIELD THEORY

Hilary Term 2016

WEDNESDAY, 13th JANUARY 2016, 9:30am to 12:30pm

You should submit answers to three out of the four questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. Consider the following Lagrangian density for an interacting real scalar field

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi(x)\partial^\mu\phi(x) - \frac{1}{2}m^2\phi^2(x) - \frac{\lambda}{4!}\phi^4(x). \qquad (*)$$

Under a dilation transformation with parameter α, space-time coordinates and the field $\phi(x)$ transform as

$$x^\mu \to x'^\mu = e^\alpha x^\mu,$$

$$\phi(x) \to \phi'(x') = e^{-\alpha}\phi(x).$$

The Noether current associated to a continuous symmetry is given by

$$J^\mu = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\frac{\delta\phi}{\delta\alpha} - J_0^\mu\,, \qquad \text{with } \delta\mathcal{L} = \delta\alpha\,\partial_\mu J_0^\mu\,.$$

(a) [4 marks] Obtain the equation of motion for the field $\phi(x)$.

(b) [6 marks] The canonical energy-momentum tensor T^μ_ν is defined as the Noether current for space-time translations. Show that it takes the form

$$T^\mu_\nu = \frac{\delta\mathcal{L}}{\delta(\partial_\mu\phi)}\partial_\nu\phi - \delta^\mu_\nu\mathcal{L}\,,$$

and calculate it explicitly for the Lagrangian density $(*)$.

(c) [6 marks] Consider a new symmetric energy-momentum tensor according to the following definition

$$\Omega^{\mu\nu} = T^{\mu\nu} + C(\partial^\mu\partial^\nu - \eta^{\mu\nu}\partial^\rho\partial_\rho)\phi^2(x).$$

Determine the constant C such that, for the massless case $m = 0$, the traceless condition holds

$$\eta_{\mu\nu}\Omega^{\mu\nu} = \Omega^\mu_\mu = 0\,.$$

(d) [9 marks] For the massless case $m = 0$, show that the four-dimensional action

$$S = \int \mathrm{d}^4x\,\mathcal{L}$$

is invariant under the dilation transformation. Calculate the Noether current $j^\mu(x)$ associated to the dilation transformation and show that it is conserved.

2. Consider a free real scalar field, in the Schroedinger picture,

$$\phi(\vec{x}) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} + a^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} \right),$$

$$\pi(\vec{x}) = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} (-i) \sqrt{\frac{E_{\vec{p}}}{2}} \left(a_{\vec{p}}\, e^{i\vec{p}\cdot\vec{x}} - a^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} \right),$$

where $\pi(\vec{x})$ is the momentum conjugate to $\phi(\vec{x})$ and

$$\left[a_{\vec{p}}, a^\dagger_{\vec{q}}\right] = (2\pi)^3 \delta^3(\vec{p} - \vec{q}), \qquad \left[a_{\vec{p}}, a_{\vec{q}}\right] = 0, \qquad \left[a^\dagger_{\vec{p}}, a^\dagger_{\vec{q}}\right] = 0.$$

(a) [5 marks] Use the commutation relations for the creation and annihilation operators to deduce the canonical commutation relation for $[\phi(\vec{x}), \pi(\vec{y})]$.

(b) [5 marks] Describe how states in the free-field Hilbert space are built using $a_{\vec{p}}$ and $a^\dagger_{\vec{p}}$. In particular, define the vacuum $|0\rangle$, single-particle $|\vec{p}\rangle$ and n-particle $|\vec{p}_1, \ldots, \vec{p}_n\rangle$ states. Show that the particles are bosons, namely

$$|\vec{p}_1, \ldots, \vec{p}_n\rangle = |\vec{p}_{\sigma(1)}, \ldots, \vec{p}_{\sigma(n)}\rangle,$$

for any permutation σ.

(c) [4 marks] The Hamiltonian and momentum operators, after normal ordering, can be written in the form

$$H = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} E_{\vec{p}}\, a^\dagger_{\vec{p}} a_{\vec{p}},$$

$$\vec{P} = \int \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3} \vec{p}\, a^\dagger_{\vec{p}} a_{\vec{p}}.$$

Determine from these the energy and momentum of the states $|0\rangle$ and $|\vec{k}\rangle$.

(d) [5 marks] Show that one can normalise the one-particle states such that

$$\langle \vec{p} | \vec{p}\,' \rangle = 2E_{\vec{p}}\, (2\pi)^3 \delta^3(\vec{p} - \vec{p}\,').$$

Show that the right hand side is Lorentz invariant.

(e) [6 marks] Write down the expansion of a scalar field in the Heisenberg picture in terms of creation and annihilation operators and show that the Feynman propagator

$$\Delta_F(x - y) = \langle 0 | T\{\phi(x)\phi(y)\} | 0 \rangle$$

is a Green's function of the Klein-Gordon equation.

3. Consider a scalar field theory with the cubic interaction in the D-dimensional Minkowski space. The action of the theory is given by

$$S = \int \mathrm{d}^D x \left(\frac{1}{2} \partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{3!} \lambda \, \phi^3 \right).$$

The N-point correlation function is defined using path integrals as

$$\langle \phi(y_1) \ldots \phi(y_N) \rangle = \frac{\int \mathcal{D}\phi \; \phi(y_1) \ldots \phi(y_N) e^{iS}}{\int \mathcal{D}\phi \, e^{iS}}. \qquad (**)$$

(a) [4 marks] Assuming that the action is a dimensionless quantity, find the mass dimension of the field ϕ and the coupling constant λ in D dimensions. Find the critical dimension D_c for which the coupling constant λ is dimensionless. What is the mass dimension of ϕ in D_c dimensions?

(b) [7 marks] Decompose the action into the free-field and the interacting part and write e^{iS} as a power expansion in λ. Write the numerator of the N-point correlator $(**)$ as a sum of free-theory correlators. Using Wick's theorem evaluate the numerator of the one-point correlator up to the order λ^2. Do not attempt to perform space-time integrals. Give a diagrammatic representation of various terms in the expansion.

(c) [8 marks] Using Wick's theorem evaluate $\langle \phi(y_1)\phi(y_2) \rangle$ up to the order λ^2. Do not attempt to perform space-time integrals. Give a diagrammatic representation of various terms in the expansion.

(d) [6 marks] Draw all connected diagrams contributing to $\langle \phi(y_1)\phi(y_2)\phi(y_3) \rangle$ up to the order λ^3 and determine their symmetry factors.

4. Consider a scalar field theory in the D-dimensional Minkowski space with Lagrangian density

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}m^2\phi^2 - \frac{1}{4!}\lambda_0\,\phi^4\,.$$

(a) [4 marks] Write down the momentum-space Feynman rules for this theory. Draw the one loop connected one particle irreducible Feynman graphs with two and four external lines.

(b) [7 marks] Show that, for k_μ a D-dimensional vector

$$\frac{1}{(2\pi)^D}\int \mathrm{d}^D k \frac{1}{(k^2+m^2)^n} = \frac{1}{(n-1)!}\frac{1}{(4\pi)^{\frac{D}{2}}}\,\Gamma\left(n-\frac{D}{2}\right)(m^2)^{\frac{D}{2}-n}\,.$$

Using this verify that

$$\frac{1}{(2\pi)^D}\int \mathrm{d}^D k \frac{1}{(k^2+m^2)((p-k)^2+m^2)} \sim \frac{1}{8\pi^2\epsilon} \qquad \text{as} \qquad \epsilon = 4-D \to 0\,.$$

The Gamma function is defined for $\Re\alpha > 0$ *by*

$$\Gamma(\alpha) = \int_0^\infty \mathrm{d}s\, s^{\alpha-1}e^{-s}$$

and satisfies $\Gamma(\alpha+1) = \alpha\Gamma(\alpha)$ *with* $\Gamma(1) = 1$. *You may also wish to use the formulae*

$$\int_{-\infty}^\infty \mathrm{d}z\, e^{-z^2} = \sqrt{\pi}\,, \qquad \int_0^\infty \mathrm{d}s\, s^{\alpha-1}e^{-bs} = \frac{\Gamma(\alpha)}{b^\alpha}\,, \qquad \int_0^1 \mathrm{d}z \frac{1}{(zA+(1-z)B)^2} = \frac{1}{AB}.$$

(c) [10 marks] Show that we may cancel the one loop divergences which arise for $\epsilon \to 0$ in the two- and four-point functions by adding to $\mathcal{L}$ the following counter-term

$$\mathcal{L}_{c.t} = -\frac{1}{2}\frac{\lambda_0 m^2}{16\pi^2\epsilon}\phi^2 - \frac{1}{4!}\frac{3\lambda_0^2}{16\pi^2\epsilon}\phi^4\,.$$

(d) [4 marks] Define the dimensionless renormalised coupling g and determine the function

$$\hat{\beta}(g) \equiv \mu\frac{\mathrm{d}g}{\mathrm{d}\mu}\bigg|_{\lambda_0}$$

to one-loop, as a function of g.