A15283W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

QUANTUM CONDENSED MATTER PHYSICS II

Trinity Term 2016

THURSDAY, 21 APRIL 2016, 14.30 to 15.30

You should submit answers to both of the two questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. In this question we consider the Ginzburg-Landau free energy for a spinless bose superfluid in D dimensions of the form

$$F = \int \mathbf{dx} \left\{ \kappa |\nabla \psi(\mathbf{x})|^2 + a(T - T_c)|\psi(\mathbf{x})|^2 + \alpha_4 |\psi(\mathbf{x})|^4 + \alpha_6 |\psi(\mathbf{x})|^6 \right\}$$

Here, T is temperature and for all of this problem you may assume $T < T_c$. You may assume the coefficients κ, a, α_4, T_c and α_6 are constants independent of temperature. You may assume a, κ and T_c are all positive. For parts (b)–(e) of this question you may assume $\alpha_6 = 0$ and $\alpha_4 > 0$.

(a) [4 marks] ψ is a complex order parameter. Briefly explain why this is appropriate for a bose superfluid.

(b) [4 marks] Determine the value of $|\psi|$ in the ground state if the system is translationally invariant (i.e., you may assume periodic boundary conditions). Call this value ψ_0. Determine the free energy per unit volume of the system in the ground state.

(c) [4 marks] Using functional differentiation (or otherwise) derive from the free energy a non-linear time-independent Schrödinger equation (otherwise known as a Gross-Pitaevskii equation).

(d) [4 marks] Define a new field $f = \psi/\psi_0$ and rewrite the above-derived non-linear Schrödinger equation in terms of this new field. Using this equation or otherwise, determine the characteristic length scale ξ, known as the coherence length. Determine how ξ depends on the temperature.

We will assume that the correlation length for fluctuations of the ψ field is given by the coherence length ξ calculated above in part (d). The Ginzburg criterion states that mean field theory is valid near the critical point when the magnitude of the mean field free energy calculated in Landau theory is much greater than the magnitude of the expected fluctuations of the free energy.

(e) [4 marks] Assume that the magnitude of the fluctuations of the free energy within a region of volume ξ^D is given by $k_b T$ with k_b being Boltzmann's constant. Comparing this quantity to the free energy per unit volume calculated above in part (b), determine the upper critical dimension D_c such that the Ginzburg criterion is satisfied for all $D > D_c$ whenever T is sufficiently close to T_c.

(f) [5 marks] By mixing certain impurities into a bose superfluid, one can tune the value of the parameter α_4. Assume that α_4 has been carefully tuned to zero for all temperatures, and assume that $\alpha_6 > 0$. Following the same reasoning as above, deterimine the value of the upper critical dimension D_c in this case.

2. Consider a three-dimensional gas of electrons in the absence of disorder.

(a) [5 marks] Neglecting the interaction between electrons, use Drude theory or simple classical mechanics to derive the finite frequency conductivity

$$\sigma(\omega) = \frac{ne^2}{-i\omega m.} \tag{1}$$

where n is the number density of electrons, $-e$ is the electron charge and m is the electron mass. This result is accurate in the limit of small wavevector $\mathbf{q}$ and large frequency $\hbar\omega \gg |\mathbf{q}|v_F$ with v_F the fermi velocity.

(b) [5 marks] Let the density-density response function $\chi(\mathbf{q}, \omega)$ be defined in terms of the *charge* density perturbtion $\delta\rho(\mathbf{q}, \omega)$ that results from an externally applied electrostatic potential $\phi(\mathbf{q}, \omega)$ where $\mathbf{q}$ is the wavevector and ω is the frequency. In particular let

$$\delta\rho(\mathbf{q}, \omega) = \chi(\mathbf{q}, \omega)\phi(\mathbf{q}, \omega). \tag{2}$$

For a non-interacting electron gas we usually call this quantity $\chi^0(\mathbf{q}, \omega)$. By using current conservation and the relationship between electric field and electrostatic potential, use equation (1) to derive the density-density response function of the non-interacting electron gas in the limit of $\hbar\omega \gg |\mathbf{q}|v_F$.

(c) [10 marks] Now consider including Coulomb interactions between the electrons. The Fourier transform of the Coulomb interaction $V(r) = 1/(4\pi\epsilon_0|\mathbf{r}|)$ is given by

$$V_q = \frac{1}{\epsilon_0|\mathbf{q}|^2}.$$

Use $\chi^0(\mathbf{q}, \omega)$ calculated in part (b) in a self-consistent Hartree (RPA or finite frequency Thomas-Fermi) approximation to determine an expression for the *interacting* density-density response $\chi(\mathbf{q}, \omega)$ in the limit $\hbar\omega \gg |\mathbf{q}|v_F$. Find the pole in this response function to determine the plasma frequency.

(d) [5 marks] Solve the classical equations of motion for a system of electrons to re-derive the plasma frequency you just derived in part (c) above. Hint: When charge in a solid is uniformly displaced from its neutralizing background it builds up charge on the edges of the solid like a capacitor.