# Examiners' Report: Final Honour School of Mathematical and Theoretical Physics Part C and MSc in Mathematical and Theoretical Physics Trinity Term 2020

March 8, 2021

## Part I

### A. STATISTICS

- **Numbers and percentages in each class.**

  See Table 1.

| | Numbers | | | | | Percentages % | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 2020 | (2019) | (2018) | (2017) | (2016) | 2020 | (2019) | (2018) | (2017) | (2016) |
| Distinction | 42 | (40) | (25) | (31) | (18) | 76 | (76) | (60) | (76) | (86) |
| Merit | 9 | (6) | (n/a) | (n/a) | (n/a) | 17 | (11) | (n/a) | (n/a) | (n/a) |
| Pass | 3 | (6) | (17) | (10) | (3) | 5 | (11) | (40) | (24) | (14) |
| Fail | 1 | (1) | (0) | (0) | (0) | 2 | (2) | (0) | (0) | (0) |
| Total | 55 | (53) | (42) | (41) | (21) | 100 | (100) | (100) | (100) | (100) |

Table 1: Numbers and percentages in each class

- **Numbers of vivas and effects of vivas on classes of result.**
  No vivas were held.

- **Marking of scripts.**
  All dissertations and three mini-project subjects were double-marked, after which the two markers consulted in order to agree a mark between them.

  All written examinations and take-home exams were single-marked according to carefully checked model solutions and a pre-defined marking scheme which was closely adhered to. One mini project subject which followed a mark scheme (Galactic and Planetary Dynamics) was also marked in the same way. A comprehensive independent checking procedure is also followed.

## B. New examining methods and procedures

Due to the pandemic, procedures were introduced. Written examinations in Trinity term took place in the form of timed open-book examinations, where students had the same length of time to complete the open-book examination as they would have had for a written examination, plus an extra hour to upload/download the examination paper, and to scan and submit their solutions. Students took the open-book examinations according to their time-zone and were required to uphold an honour code. Oral presentations were also cancelled. Candidates had the option to request that they be awarded Declared to Deserved Masters (DDM) instead of a Distinction, Merit, Pass or Fail classification. Candidates were permitted to drop one formally assessed unit. A safety net was also implemented where the number of completed units required to achieve each classification available was reduced by one.

## C. Changes in examining methods and procedures currently under discussion or contemplated for the future

None.

## D. Notice of examination conventions for candidates

Notices to candidates were sent on: 15 October 2019 (first notice), 15 November 2019 (second notice), 28 February 2020 (third notice) and the 13 May 2020 (final notice).

The examination conventions for 2019-2020 are on-line at http://mmathphys.physics.ox.ac.uk/students.

# Part II

## A. General Comments on the Examination

## B. Equality and Diversity issues and breakdown of the results by gender

*Removed from public version*

**Oral Presentation** Oral presentations were cancelled due to the pandemic in 2019-20, and were not a requirement to pass the degree.

## C. Detailed numbers on candidates' performance in each part of the examination

The number of candidates taking each paper is shown in Table 2 and in the Average USM per Formal Assessment graph below. In accordance with University guidelines, statistics are not given for papers where the number of candidates was five or fewer.

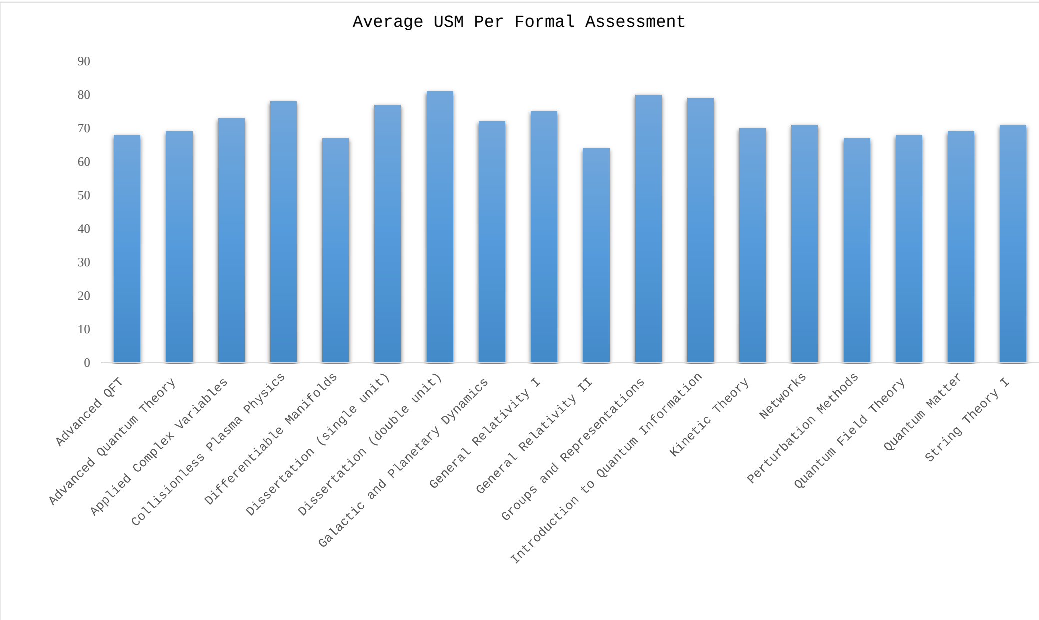

Table 2: Numbers taking each paper

| Paper | Number of Candidates | Avg USM | StDev USM |
|---|---|---|---|
| Advanced Fluid Dynamics | - | - | - |
| Advanced Philosophy of Physics | - | - | - |
| Advanced Quantum Field Theory | 28 | 68 | 16.8 |
| Advanced Quantum Theory | 22 | 69 | 13.07 |
| Algebraic Geometry | - | - | - |
| Algebraic Topology | - | - | - |
| Applied Complex Variables | 8 | 73 | 12.96 |
| Category Theory | - | - | - |
| Collisionless Plasma Physics | 8 | 72 | 11.41 |
| Continuous Optimisation | - | - | - |
| Differentiable Manifolds | 7 | 67 | 11.08 |
| Disc Accretion in Astrophysics | - | - | - |
| Dissertation (single unit) | 13 | 77 | 8.21 |
| Dissertation (double unit) | 16 | 81 | 7.02 |
| Finite Element Methods for PDEs | - | - | - |
| Galactic and Planetary Dynamics | 6 | 72 | 10.03 |
| General Relativity I | 26 | 75 | 15.18 |
| General Relativity II | 12 | 64 | 17.88 |
| Geometric Group Theory | - | - | - |
| Groups and Representations | 31 | 80 | 12.49 |
| Introduction to Quantum Information | 30 | 79 | 14.99 |
| Kinetic Theory | 16 | 70 | 17.14 |
| Lie Groups | - | - | - |
| Networks | 11 | 71 | 9.98 |
| Numerical Linear Algebra | - | - | - |
| Perturbation Methods | 10 | 67 | 11.03 |
| Quantum Field Theory | 46 | 68 | 12.89 |
| Quantum Matter | 12 | 69 | 20.86 |
| Radiative Processes and High Eng. Astro. | - | - | - |
| String Theory I | 16 | 71 | 5.19 |
| Supersymmetry and Supergravity | - | - | - |
| Theories of Deep Learning | - | - | - |

The number of candidates taking each homework completion course is shown in Table 3. In accordance with University guidelines, statistics are not given for papers where the number of candidates was five or fewer.

Table 3: Numbers taking each homework completion course

| Paper | Number of Candidates | Percentage completing course |
|---|---|---|
| Advanced Fluid Dynamics | - | - |
| Advanced Quantum Theory | - | - |
| Advanced Supersymmetry | - | - |
| Astroparticle Physics | - | - |
| Collisionless Plasma Physics | - | - |
| Conformal Field Theory | 9 | 100 |
| Cosmology | 12 | 100 |
| Disc Accretion in Astrophysics | - | - |
| Galactic and Planetary Dynamics | - | - |
| Group and Representations | 31 | 100 |
| High Energy Density Physics | - | - |
| Kinetic Theory | - | - |
| Nonequilibrium Statistical Physics | 16 | 100 |
| Quantum Field Theory in Curved Space Time | 12 | 83 |
| Quantum Matter | - | - |
| Renormalisation Group | 13 | 100 |
| Soft Matter Physics | 9 | 100 |
| String Theory II | - | - |
| Supersymmetry and Supergravity | 12 | 100 |
| Symbolic, Numerical and Graphical Scientific Programming | 10 | 100 |
| The Standard Model and Beyond I | - | - |
| The Standard Model and Beyond II | - | - |
| Topics in Soft and Active Matter Physics | - | - |
| Topological Quantum Theory | 24 | 100 |

## D. Assessors' comments on sections and on individual questions

### Advanced Fluid Dynamics

Question 1: Q1 proved relatively easy, although, as usual, not as easy I thought it would be.

Part (a) was pure bookwork, but, even in open-book format, it proved surprisingly challenging for most students to lay out with clarity what the iMHD equations were and what the assumptions behind them were. Typical omissions included not stating the equation for $\tilde{p}$ (needed to enforce incompressibility) or not realising that $\beta$ and Ma were related in the iMHD ordering. In a couple of cases, the students were unclear about the difference between iMHD and MHD.

Part (b) was done well by all, although not all were 100% clear about the logic of how one showed that 2D *implied* $B_x = \partial_y A$ and $B_y = -\partial_x A$, rather than merely that the latter was a consistent choice.

Part (c): everyone got full marks for getting to the result correctly, although most did it by a longer route (via vector algebra in 3D) than necessary, not realising that $B^2 = (\partial_x A)^2 + (\partial_y A)^2$, varying which would have been really easy.

Part (d): no one realised that in iMHD, $\tilde{p}$ in principle had a magnetic part because the Lorentz force is not divergence-free and that, therefore, one had to spot that the force was $-\nabla \alpha A^2/2$, which could be absorbed into $\tilde{p}$ and the latter would then be zero in the absence of flows.

Question 2: Surprisingly few candidates attempted all parts of this question. Some derivations in "show that" parts of questions had large gaps immediately before the required result.

Part (a) Some candidates showed $\nabla^2 \mathbf{u} = 0$ but did not show that $\nabla \cdot \mathbf{u} = 0$. Both calculations are simplest in index notation.

Candidates were then expected to find the flow around a rotating sphere by considering $\mathbf{c} = \boldsymbol{\Omega}$ and $\phi \propto 1/r$. Most started from scratch instead. The torque about the rotation axis is (with outward normal $\mathbf{n} = \mathbf{e}_r$)

$$T = \mathbf{k} \cdot \iint \mathbf{x} \times \boldsymbol{\sigma} \cdot \mathbf{n} \, \mathrm{d}S = \iint a \sin\theta \, \sigma_{\phi r} \, \mathrm{d}S = 2\pi a^3 \int_0^{\pi} (\sin\theta)^2 \, \sigma_{\phi r} \, \mathrm{d}\theta.$$

Several candidates lost one or more factors of $\sin\theta$ from trying to find the $\theta$ or $\phi$ component of the torque.

Part (b) The total torque on the particle must vanish in Stokes flow, so the viscous torque $\mathbf{T}$ is *minus* the gravitational torque. This gives $\mathbf{T} = h\mathbf{p} \times m\mathbf{g}$, with $\mathbf{g} = -g\mathbf{k}$ pointing downwards. Several candidates made sign errors.

Even though this was sat as an open book exam, it was still necessary to show the derivation of the contribution from $\widetilde{\mathsf{H}}\,\mathsf{E}^{\infty}$, not just assert that it was covered in lectures.

Noone stated explicitly that the torque $\mathbf{T}$ is perpendicular to $\mathbf{p}$, like the contribution from $\widetilde{\mathsf{H}}\,\mathsf{E}^{\infty}$, so $\mathsf{C}^{-1}\mathbf{T} = \mathbf{T}/Y^C$.

Few candidates addressed the special case of a sphere, and noone noticed that part (a) gives $Y^C = 8\pi a^3$ for a sphere.

Part (c) Only a few candidates attempted this part. They all found the evolution equation for $\Theta$, but noone found the steady solution for long rods with $\beta = 1$. It is best to formulate a quadratic equation for $\cos\Theta$ and consider the "$-$" root to show that a steady solution always exists. For spheres ($\beta = 0$) one can solve immediately for $\cos\Theta = -\Omega/\Gamma$, so a steady solution only exists when $|\Omega| \leq \Gamma$.

**Advanced Quantum Field Theory**

All questions were to be attempted. The standard of answers was good in general. The average mark for question 1 one was somewhat higher than for questions 2 and 3, I suspect because this contained a greater deal of bookwork, and also because the majority of students will have started with this, hence making any timing issues less significant here. Further comments follow below.

**Question 1)**. a) i) was basic bookwork, and all students got this right. ii) contained some fairly basic manipulations and again almost all students had no issue here, with the same being true for the Feynman diagram in part iii) and the result in iv). Part v) was unseen, and while some students struggled to get every element of the manipulations correct, most managed at least part if not all of the required work. Most students spotted that the integral in vi) was finite. Part vii) was unseen, though the answer was contained in the lecture notes; many students managed to get the answer via a correct method, though not all. Part viii) was bookwork and almost all students managed this.

**Question 2)**. a) i) was bookwork, and almost students got full marks. ii) was unseen, and while many students managed to get full or close to full marks, a non–negligible minority struggled to solve the question at all, or else did not have time to. Part iii) was unseen, and the students were evenly divided between those who could spot the answer, getting full marks, and those who could not. b) i) was unseen, but almost all students could identify the correct diagrams. ii) was unseen, and a large fraction of students appeared to struggle with the required manipulations of Dirac algebra, at least given time constraints; a sizeable minority managed to get full marks, however. iii) was unseen, and more challenging. Only a small fraction of students managed to get full marks here, with many struggling or running out of time.

**Question 3)**. a) i) was bookwork, and almost all students got full marks. ii) was unseen, and was fairly evenly divided between students would managed to get very high marks, those who managed to get part of the answer out, and those who struggled to approach the question entirely. b) i) to iii) was a mixture of bookwork and unseen material, and most students managed to get high marks. iv) involved a simple observation, which almost all students managed. v) was unseen, and designed to be challenging.

**Advanced Quantum Theory**

**Q1** This question was on the use of transfer matrix to solve an Ising model with vacancies. Part (a) was successfully completed by the vast majority of students as it was a modest generalization of problems from the notes, the homework, past papers, and revisions. Part (b) was similarly a straightforward exercise, but two common errors were failing to explain why it was always possible to diagnolize the transfer matrix (because it is real and symmetric) and, although the eigenvalues were given to the students, identifying the maximal eigenvalue tripped some students up. Part (c) was to obtain a formula for the average number of vacancies and examine it in a limiting case of very high temperature. Despite being given a generous hint to try to express this in terms of the free energy (by use of standard ideas from undergraduate statistical mechanics, it can be written as $-\partial f/\partial\mu$ with $T$ fixed) some student stumbled, with a very small number inexplicably trying to compute powers of the 00 component of the transfer matrix). Most students who followed the hint successfully obtained the answer. Part (d) involved computing the entropy per spin in a different limit, where the vacancy chemical potential is driven to $-\infty$. Full points were only given if the correct physical point was made, namely that in this limit the model reduces to the standard Ising model — something missed by a surprisingly large number of students.

**Q2** This question concerned application of Holstein-Primakoff mapping and Bogoliubov transformations to solve for the spin wave spectrum of an easy-plane XXZ magnet. Much of this was straight bookwork, and actually (in slightly different notation) was covered in the revision classes. Parts (a, b) focused on classical ground states; most students successfully attempted these, though the explanations of spontaneous symmetry breaking were occasionally spotty. Part (c) was a written answer on the nature, usefulness, and complications of Holstein-Primakoff transformations. Many students struggled to give cogent and complete answers here, with only a handful providing enough information to obtain the full 4 marks. Parts (d) and (e) involved implementing the Holstein-Primakoff expansion to find the linear spin-wave Hamiltonian, and finding its Fourier transformation. The overwhelming majority of students scored full marks on part (d), and *every* student completed part (e) perfectly (though this is unsurprising, since the answer was stated in the question). The final section, part (f) consisted of implementing the Bogoliubov transformation. Leniency was exercised even for not-quite-correct choices of the Bogoliubov transformation, but many students stumbled when extracting the spin-wave dispersion. A key mistake (surprisingly) was in sloppy Taylor expansions: students neglected to keep track of $O(k^2)$ terms in the expansion of $\cos^2 k$. This led to erroneous results for the spin-wave speed $v_s$ for $\Delta < 1$, but in many cases such mistakes could have been easily caught by observing that the correct answer for $v_s$ must vanish as $\Delta \to 1^-$.

**Collisionless Plasma Physics**

The standard of the answers in this exam were generally high. All questions were to be attempted. The difficulty of the questions was set at the correct level.

Question 1.

In part a) some candidates forgot to supply the requested sketches, or drew them incorrectly. The proofs in b) and c) were generally completed well. In part d) some candidates were not able to find $U_{\min}$ and $U_{\max}$ correctly. In part e) those candidates who had difficulties in part d) struggled to give clear answers. Finally, in part f) most candidates were able to find the distribution function, although most candidates did not attempt to use the method of characteristics, but instead opted to find and solve an ODE for $f(z)$. Most candidates failed to notice that $\mu_{\min}$ must be positive definite to avoid unphysical divergences in integral for the particle density.

Question 2

All candidates found the correct result in part a). Part b) was largely bookwork. In part c) most errors stemmed from not keeping track of the sign of $\omega'_{*,s}$. Few candidates realised that the drift wave propagates in the $-\hat{\mathbf{z}}$ direction. The proof in part d) was bookwork. Some candidates lost marks because of sign errors, and an incorrect description of the $\eta \gg 1$ expansion. Part e) was generally answered poorly, with few candidates obtaining the correct result. Some candidates made the error of using an inconsistent dominant balance to solve the polynomial equation, although the most common errors were in manipulation. Finally, few candidates were able to obtain a fluid equation for both the temperature and the density in part f), and only limited physical descriptions were given.

Question 3

Part a) was bookwork. In part b) some candidates made the unjustified assumption that $\omega_R \gg \omega_{UH}, \omega_L$, and hence, they did not calculate the wave number correctly. In part c) the most common errors were a lack of discussion of the phase of the wave amplitude, misidentifying the reason for the undetermined constant in the solution, and failing to give a correct derivation for the modulus of the wave amplitude. Part d) was generally completed with only minor errors. Part e) was bookwork, with several candidates performing the matching correctly. However, some candidates made an error of sign in defining the boundary layer length scale. Finally, in part f) only some candidates could argue why $\omega > \omega_R$ would be desirable for experimental measurements using X-modes.

**Disc Accretion in Astrophysics: Theories and Applications**

The results seem quite high; this is not because the exam was too easy, but because all candidates deserved such high marks.

There is one unfortunate error in question 2 (and the solution), the second term in the solution given to the second part should have a "3" in the numerator, not a "9". All the students got this right, but probably wasted some time trying to find a mistake they did not make.

**Galactic and Planetary Dynamics**

This project was an investigation of mean-motion resonances in the restricted 3-body problem. Parts (i)–(iii) were essentially bookwork and were answered very well. Part (iv) involved applying this bookwork to construct an averaged Hamiltonian and, again, candidates did well here. The numerical calculations in Part (v) proved more challenging though. Most candidates could construct a decent integrator, if only on paper. Only the very best could use their integrator to demonstrate that the averaged Hamiltonian found in Part (iv) does provide a good description of orbits near the 3:2 resonance of the original Hamiltonian.

**Groups and Representations**

**Question 1:** This was a question testing some basic knowledge on representations and featuring applications of Schur's Lemma. It was tackled by 21 students and with an average of 21.7 out of 25 marks was completed extremely well.

**Question 2:** In this question students had to work out the representation theory of the group $S_3$, given as a two-dimensional matrix representation, and apply their results to a couple of given representations. This question was chosen by all students and was completed extremely well. The few problems which arose were usually related to part (d), where some had problems extracting the irreducible representation content correctly and struggled to find the singlets.

**Question 3:** This question dealt with the group $SU(3) \times SU(3)$ and its subgroup $SU(3) \times SU(2)$ in view of a possible unification group for the standard model. Main difficulties were to extract the correct $U(1)$ and compute its charges in part (d) and to carry out a meaningful discussion about the possible usefulness as a unifying group in part (e).

**Question 4:** This question was exploring the representation theory of $SU(4)$ and the possible use of this group as a quark flavour symmetry. No particular part of the question caused excessive problems and loss of marks usually resulted from smaller problems, such as identifying the correct basis of generators in part (a), finding all the correct representations and their associated tensors in part (b) and coming up with a fully satisfactory interpretation of results in parts (d) and (e).

### Kinetic Theory

Question 1: The overall standard was high. Several candidates produced complete or near-complete solutions.

Part (a) was mostly done well. A few candidates forgot that the temperature is the moment with respect to $|\mathbf{v} - \mathbf{u}|^2$ rather than $|\mathbf{v}|^2$, and a few omitted the $1/\rho$ and $1/(3\rho)$ normalising factors.

Part (b) is tackled most simply by multiplying the Boltzmann equation by $1 + \log f$, since $\partial_t(f \log f) = (1 + \log f)\partial_t f$, and similarly for $\nabla(f \log f)$. Several candidates multiplied by $\log f$, then either used the mass conservation equation or simply lost the remaining terms. The result $S \leq 0$ holds for all $f$, but a few candidates expanded for $f$ close to $f_0$. The simplest approach observes that $\log f_0$ is a sum of collision invariants (a few candidates incorrectly wrote that $\log f_0$ is proportional to $|\mathbf{v}|^2$) and that $(\log f - \log f_0)(f - f_0) \geq 0$.

Part (c) Several candidates wrote down mass and momentum conservation equations with unevaluated integrals containing the Lorentz force. Deriving the mass conservation equation motivates writing the Lorentz force term as the divergence $\nabla_{\mathbf{v}} \cdot ((\mathbf{E} + \mathbf{v} \times \mathbf{B})f)$. This simplifies the derivation of the momentum and momentum flux evolution equations. Some candidates lost terms by trying to integrate by parts in vector (dyadic) notation, then reinstated the missing terms at the last step to match the given result.

Part (d) Many candidates incorrectly asserted that $Q_{ijk}$ vanishes, not that $\partial_k Q_{ijk}$ vanishes because $Q_{ijk}$ is spatially uniform (as $f$ is). Several candidates just wrote down the displayed equation, with no attempt to evaluate the $\Pi_{ik}\epsilon_{kjl}B_l$ term and its transpose for the given $\mathbf{B}$. A few candidates calculated the eigenvalues of the displayed $3\times3$ matrix containing components of $\mathsf{T}$. Instead, one first needs to formulate a matrix equation with the components of $\mathsf{T}$ as the right-hand side. The displayed matrix has no $zz$ component and zero trace, so $T_{zz}$ and $T_{xx}+T_{yy}$ both decouple and decay like $\exp(-t/\tau)$. Several candidates made these observations, perhaps inspired by thinking about conservation of energy under collisions. The remaining system for $T_{xy}$ and $N = T_{xx} - T_{yy}$ has separable solutions proportional to $\exp(\sigma t)$ with $\sigma = -1/\tau \pm 2iB$. For spatially uniform solutions, the mass and momentum conservation equations reduce to $\partial_t\rho = 0$ and $\partial_t(\rho \mathrm{u}) = \rho \mathrm{u}\times\mathbf{B}$. Both are compatible with $\mathbf{u} = 0$.

Question 2: Question 2 appears to have challenged the students reasonably.

(a) Nearly everyone understood that the way to get these equations was by taking moments of the kinetic equation.

(b) Nearly everyone understood that electron inertia could be neglected. Some, not all, also realised that multiplying by $v$ and integrating would establish the balance between electric force and electron pressure gradient. There was some confusion about whether the key distinguishing feature between ions and electrons was that ions were assumed cold or that electrons had smaller mass and streamed quickly across perturbations (the latter of course).

(c) Quite a few students failed to see that using $E = -\partial\varphi/\partial x$ led to the equation for $\delta f_e$ being instantly integrable, $\delta f_e = e\varphi f_{0e}/T_e$, and that to prove isothermality of the electrons all one needed to do was take the density and pressure moments of this result. There were some unnecessarily complicated derivations, some of them partially wrong.

(d) Surprisingly few students realised that the Poisson equation coupled with the assumption $k\lambda_{De} \ll 1$ was the way to prove that $\delta n_e = Z\delta n_i$. Most figured out though how the wave equation was supposed to emerge and that the waves it described were sound waves.

(e) Very few provided a clear and crisp answer to this question. The substantive issue was the lack of clarity that there were two types of Landau damping, on electrons and on ions, that the electron damping was due to very slow electrons and so neglected by the assumption that $v \sim v_{the}$ in the electron equation, that the ion damping was neglected when $\delta p_i$ was dropped, on account of waves being much faster than the ions, and that the latter approximation depended on the assumption $T_e \gg T_i$, with heavy damping expected otherwise. Some students offered irrelevant regurgitation of lecture material about the ion acoustic instability.

Question 3: The standard was pleasantly high, Parts(a)–(c) were very well answered, followed by graceful degradation as they moved from sections (d) to (f).

**Quantum Field Theory**

Question 1. This question was well done. Most candidates knew the Feynman Rules and were able to apply them to the scattering process. The most common errors were to omit a possible channel in the fermion-fermion and fermion-antifermion scattering processes, and to omit to keep track of spinor indices in calculating $|M|^2$. There were many good attempts at the last part of the question on naive scaling dimension; a few candidates resorted to producing formulae from memory without justification.

Question 2. The first part of the question was very well done with almost all candidates able to calculate the required commutators. However very few candidates realised that to demonstrate the operation of $C_{\frac{\pi}{2}}$ one should just set $s = \pm 1$ in the results already derived which was disappointing. There were a few good attempts at deriving the Parity operator in the last part, but overall candidates found this question hardest of the three.

Question 3. This question was well done. Most candidates did a good job of demonstrating the required results in the first part. The explanations of the origin of combinatorial factors for Feynman diagrams were less secure. Many candidates clearly in fact understood this well and were able to calculate the factors in the last part in a manifestly correct way. However some candidates failed to show clearly how they had actually computed the factor claimed, and some just quoted the answers.

**Quantum Matter**

This exam worked fairly well, with a wide distribution.

The first question was the easier one. Almost all of the students were able to do parts a-c near perfectly. Part d is a generalization of what we did for homework, and about half of the students were able to handle it perfectly. On part (e) while many students knew roughly what to do, no one managed to solve the problem completely, despite the fact that it is only two lines of calculation! (Perhaps because this was less familiar territory).

The second question was harder. The earlier parts should have been fairly easy but did give some people trouble. The question was very similar to one of the homework problems. A fair number of students did not manage to correctly solve the noninteracting electron problem in part (b). Many students did not realize that a spin triplet can have $Sz = 0$, getting the last part of (b) wrong. For (c) there were a lot of errors of factors of 2 related to overcounting Hartree-Fock energy (Koopmans' factor of 2). I expected part (d) to be difficult, but it was nice to see that several students set it up correctly.

**Radiative Processes and High Energy Astrophysics**

**Radiative Processes**

This question was generally answered very well. A few candidates made typographical errors when writing down the general form of the equation of statistical equilibrium. A few mistakes were made in the final part of the question, which involved deriving a formula for the flux ratio of two emission lines, correcting the observed ratio for dust extinction, and finally re-arranging to measure the temperature of the observed nebula. The two mistakes made were: 1) incorrect formula for $C_{13}/C_{12}$ (wrong statistical weights in the expression), and 2) incorrect treatment of dust extinction (essentially assuming that magnitude relates to flux as $m_\lambda = -\ln[F_\lambda/F_\lambda^{\rm ref}]$ instead of $m_\lambda = -2.5\log_{10}[F_\lambda/F_\lambda^{\rm ref}]$, which introduces a small error).

**High Energy Astrophysics**

This was generally answered very well. Every candidate was able to derive the correct expressions. A few marks were dropped due to the final numerical answer being incorrect. Candidates struggled most with the final part of the question. Most recognised that the inclusion of general relativity would bring in both gravitational redshift and light bending, and most recognised that the iron line profile should be the same for AGN and XRBs, although there was the odd mistake. The point that no candidate got: we assume the disc to be radiation pressure dominated, which is fine for $L = 0.5L_{\rm Edd}$, but is not at all a good approximation for $L = 10^{-5}L_{\rm Edd}$ (gas pressure will likely dominate in this regime). However partial credit was given for those who mentioned that a thin disc may not be present for such a low luminosity.

There was, unfortunately, a typo in the question (my apologies). Candidates were asked to show that the maximum Doppler shift is

$$\delta_{\max} = \frac{\sqrt{1 - r_g/r}}{1 - (r/r_g)\sin i}. \tag{1}$$

However, the correct expression is

$$\delta_{\max} = \frac{\sqrt{1 - r_g/r}}{1 - (r_g/r)^{1/2}\sin i}. \tag{2}$$

Fortunately, every candidate derived the correct answer, and so the mistake did not have any effect on the final marks. There was a query about the formula in question, but this corresponded to the numerator, in which there was no mistake.

The'correct' formula is still not accurate in reality, but this is a feature of the question: the equation is derived using Special Relativity. Once General Relativity is used instead (as the candidates are prompted to consider in the question), the equation is modified. Gravitational redshift means that the numerator goes from $\sqrt{1 - r_g/r}$ to $\sqrt{1 - 3r_g/r}$ (one candidate, impressively, derived exactly this from the Schwarzschild metric), and light bending modifies the denominator.

### Supersymmetry and Supergravity

The students did very well overall.

There was some confusions in Q2 d) and Q2 e). Recall that the condition to have a supersymmetric vacuum is $\frac{\partial W}{\partial \phi} = 0$ *simultaneously* for every $\phi$. The condition $\frac{\partial V}{\partial \phi} = 0$ is the condition to have a vacuum at all (supersymmetric or not), and the former condition implies the latter since $V = |\partial W|^2$.

### C3.4: Algebraic Geometry

The paper was done very well by 12 candidates.

Question 1 had 11 answers at an average of 21 marks. Candidates lost some marks on imprecise explanations of easy checks. In part (d) some candidates explained what are the closed subvarieties of $\mathbb{A}^1$ but missed identifying the irreducible ones. In part (e) some candidates thought $(0, 0)$ was a separate component; others failed to prove that their proposed components are indeed irreducible. One of the equations in (e) had an obvious misprint that was corrected by all candidates (or not even noticed).

Question 2 had 8 answers, also at an average of 21 marks. Most of those who lost marks did so on the computation in (e). There is in fact a condition missing in (e), in that it should be assumed that the quadrics define a reduced ideal; most candidates tacitly assumed this.

Question 3 had several answers lacking a full explanation in (b) of the fact that the map $\pi$ was a morphism of quasiprojective varieties. Some candidates failed to check that in (c)(i) their proposed inverse does indeed provide an inverse.

**C3.1: Algebraic Topology**

Almost all candidates picked Q1, and then about two thirds of students picked Q3 instead of Q2. The standard of almost all scripts was very high. In Q1(b) some candidates did not carefully show functoriality. In Q1(c) there is a difference in the homology and cohomology calculations, but some leniency was allowed if candidates checked the homology version of the Kunneth theorem instead of the cohomology version. Q2(c ) candidates sometimes failed to carefully explain why the cup product lands in the correct chain complex. Q2(d) requires a commutative diagram, relating the map from (c) with the usual cup product on X, and only one candidate did this correctly. In Q3(d) some candidates did not carefully justify why the preimage of a point is a finite set. Almost no candidate explained in a satisfactory way why the local degrees are all +1. They did not notice that the previous step in (c ) had built the pull-back orientation precisely so that the orientation generator of X maps to that of Y, locally.

**C5.6: Applied Complex Variables**

Q1: Part (a) was generally well done, although many candidates failed to explain why the majority of the real axis maps to straight-line segments (only explaining what happens at the pre-images of the vertices). In (b), there were quite a few difficulties with evaluating the constants $P$, $Q$ and $\lambda$. Part (c) was well done. For part (d), a number of candidates did not adequately explain their sequence of conformal mappings, which in some cases was wrong, but many got the correct mapping of the hodograph plane to the upper half plane. The last part was naturally done by those who had successfully found $\lambda$ in part (b).

Q2: Parts of this question were done well, though some struggled with part (b), and the very last part of (d) was not solved by anyone. Part (a) was fine. In part (b), quite a number of candidates assumed that $\tilde{w}_- = -\tilde{w}_+$ when calculating the integral (as for most examples seen during the course), despite having quoted the correct expressions for those two quantities. Part (c) was done easily by most candidates. In part (d), most realised they could use the result from part (b), and obtain the required result by taking $H = 0$. A few candidates had the right idea for the last part, but chose expressions for $H(z)$ that did not tend to zero at infinity as required.

Q3: This was marginally the most popular question and was done well, although the final answer was found by only a couple of candidates. There were few difficulties with parts (a), (b) and (c), although some failed to adequately explain why the given expression holds on the strip in part (c). For part (d), the Wiener-Hopf argument was explained well by many candidates, but evaluating the integral resulted in many errors keeping track of the square roots and 'i's. A few ended up with the correct expression but multiplied by a complex number (whereas the solution should clearly be real).

**C2.7 Category Theory**

There were plenty of answers demonstrating very good understanding of the material, though none was perfect.

Question 2 was the most straightforward question and there were attempts from almost all the candidates, some of which were very good though none obtained full marks. There was a minor misprint in 2 (b) (iv), where a superscript 'op' was missing in $\mathrm{Hom}_{\mathrm{Cat}}(C^{op}, \mathrm{Set})$, but it seemed clear that it did not affect any candidates adversely; indeed only one gave any indication of noticing it.

Question 1 contained a more substantial amount of unseen material. There were many very serious attempts, and each part of the question was answered correctly by several candidates, but no candidate gave a completely successful solution to the whole question. In 1(a) it was disappointing to see that, just as last year, few candidates could give an accurate construction for coequalisers in Set.

Question 3 did not attract many attempts. The bookwork part of the question was answered well, but the rest of the question, especially (b)(iv), caused difficulties.

**C6.2: Continuous Optimisation**

All questions were attempted by various students. The performance was particularly good especially on Problem 2. Problem 1 was also very accessible, with a trickier bit in part b) and incomplete answers in part c). Problem 3 was not difficult and the students did well on part a), apart from some incomplete answers arguing uniqueness or existence. The theoretical part in Problem 3 was generally incomplete, probably due to time constraints.

**C3.3: Differentiable Manifolds**

**Question 1.** Most students lost a mark on part (a) because they missed out some detail in the definitions, typically in the definition of the free and properly discontinuous action of a group by diffeomorphisms. Part (b) was mainly done well, with a few students losing marks through small slips in showing that the volume form defines an orientation. Almost all students had the right idea for (c)(i), but most commonly lost marks for not showing that the orbits of the open sets they chose were disjoint, and for not observing that $f_r(z) \in \mathcal{S}^3$ for $z \in \mathcal{S}^3$. Part (c)(ii) was similar in that most students had the right idea and approach, but lost marks through lack of details; a common error was not showing that $f_r$ is orientation-preserving correctly, and not showing that the 3-form defined on the quotient is nowhere vanishing. About half of the students who attempted this question had the right idea for (d), all using $M \cong \mathbb{RP}^2 \times \mathcal{S}^1$ as the example, but did not fully justify their answer. This was the least popular question and produced a wide spread of marks, from high to low.

**Question 2.** Part (a) was done well. A number of students dropped marks in (b), typically by not observing that the putative vector fields and/or diffeomorphism they constructed are smooth. Part (c) was mainly done very well, but students lost marks in not explaining that for a diffeomorphism the induced map on de Rham cohomology is an isomorphism. Part (d)(i) was done very well by almost all students, with marks only lost for not referring back to the criterion for parallelizability in (b) and for not checking the homotopy they construct is well-defined. Whilst most correctly students correctly understood that $\mathcal{S}^2 \times \mathbb{R}$ is parallelizable in (d)(ii), most did not justify it adequately. This was the most popular question, with all but one student attempting it. There was a spread of marks, but no low marks (below 10).

**Question 3.** Part (a) was usually done well, though some students dropped marks by not stating the criterion for a map to be a local diffeomorphism clearly. Most students dropped a mark on part (b), usually by not defining the interior product. Part (c)(i) was done well, except some students thought the flow was a vector field, even though they defined it correctly in (a). Part (c)(ii) created a great variation in responses, with most difficulties arising in the correct computation of the Lie derivative from the definition, but also in computing the exterior derivative and the interior product correctly. In part (d), most students had the right idea, but some made errors in their computations. There was a spread of marks, but also no low marks (below 10).

### C6.4: Finite Element Methods for Partial Differential Equations

Q1: This question was attempted by 38% of candidates. It revealed a good spread of abilities across those who attempted it. Q1 (b) was for the most part well answered, although some candidates neglected to consider the normal component of the gradient so as to apply the second factorisation lemma. Q1 (c) was very well answered by all who attempted it, although some candidates neglected to provide a counterexample (i.e. computed the determinant of the matrix to be zero, without exhibiting an element of the kernel). Q1 (d) was quite challenging, with no student successfully completing Q1 (d) (ii).

Q2: This question was very popular, with every candidate attempting it. Unsurprisingly, most candidates did very well in the bookwork in Q2 (a). Q2 (b) considered the vector-valued equations of linear elasticity; this was handled much better than vector-valued equations in previous examinations, and almost every student successfully integrated the grad-div term by parts, unlike in previous years. In Q2 (b) (ii), several students applied the fact about the Frobenius inner product *twice*, to replace the inner product of the symmetric gradients with the inner product of the gradients to acquire the familiar bilinear form arising in the vector Laplacian. The second application of this was erroneous. In Q2 (b) (iii), surprisingly few students used the hint to derive Young's inequality, and many failed to prove the required bound on the divergence. Q2 (c) was unseen but was generally answered well, indicating a good understanding of Céa's Lemma. Q2 (d) was also unseen, but related to the Stokes equations studied in lectures. It was quite poorly answered. A remarkable number of candidates wrote down only one equation for two unknowns; others neglected the requirement that the bilinear form be symmetric; and no candidate correctly identified the space for the Lagrange multiplier as $L^2(\Omega)$ instead of $L^2_0(\Omega)$ (which is only appropriate in the incompressible limit).

Q3: This question was attempted by 62% of candidates. The early parts of Q3 (a) were answered well, but many students struggled to correctly state the Newton–Kantorovich iteration for systems of partial differential equations in Q3 (a) (iv). Q3 (b) (i) and (ii) were bookwork and were well-answered, but Q3 (b) (iii) challenged those with a weaker grasp of the inf-sup condition. Q3 (c) was a challenging question about refining the bookwork error estimate for noncoercive problems and gave an opportunity for the best candidates to distinguish themselves.

### C3.2 Geometric Group Theory

**Question 1:** This was attempted by all candidates, with fairly complete answers provided, in particular for the examples in the second part. Only half of the candidates attempted the question where the isomorphisms between two groups was to be proven by finding a finite sequence of Tietze transformation, and that seems to show that while the formal knowledge of this method was acquired, the intuition behind it is still lacking.

**Question 2:** This question was for the main part purely theoretical. Nevertheless, while most students displayed confident knowledge of everything related to the fundamental group of a graph of groups, very few attempted the second part of the question, requesting to provide a definition of such a group for finite simplicial trees *via* a universal property.

The third part on residual finiteness was answered reasonably well.

**Question 3:** This question was attempted in equal measure as Question 2, and answered very well on the whole, including the parts requiring an algorithmical approach. Surprisingly the least well answered part was part (b), in which a simple geometric argument, relying entirely on the geometry of hyperbolic spaces, was all that was needed.

### C7.5: General Relativity I

#### Question 1

This question was the most popular of the three questions, and was generally done well by a lot of students. The bookwork parts ((a) and (b)) were almost exclusively done correctly. The new parts ((c) and (d)) caused some more problems; particularly part (d). Here, the most common issue was that students were able to correctly write down expressions for the motion of the second light signal in the form of integrals, but did not notice that, since they were only asked to produce an answer correct to $O(\Delta\tau)$, they could differentiate these expressions in $\Delta\tau$ to obtain the result. Another common mistake was, when considering the motion of the satellite, to consider its radial position as a function of its proper time (that is, $r(\tau_S)$) but to neglect the dependence of the time coordinate $t$ on the proper time $\tau_S$. This is a conceptual issue that is most likely caused by students familiarity with flat space, where one doesn't need to consider such things.

**Question 2**

This question was by far the least popular, most likely as it appears the most technical and mathematical of the three options. However, it was generally completed very successfully by the small number of students who attempted it. This question did include a typo: in part (c), the order of the arguments of $\lambda$ should have been reversed, writing $\lambda_{[a\mu+\mu',X]}$ instead of $\lambda_{[X,a\mu+\mu']}$. Fortunately this did not appear to cause any confusion (although it was commented upon by some students!), as the rest of the notation was consistent. Where mistakes were made in this question, the typical error was to believe that the commutator is $C^\infty$ linear (so $[X, aY] = a[X, Y]$ for vectors $X$, $Y$ and a scalar field $a$, which is not true) and then to compensate for this error by also failing to properly apply the Leibniz rule for vector fields.

**Question 3**

Part (a) was done very successfully, with students demonstrating knowledge of the Einstein equations and the various terms appearing in this equation. Part (b) was also bookwork, and was done correctly by many, although a fair number of students were confused regarding the difference between proper time and a general affine parameter. Part (c) was done very successfully, and part (d) caused by far the most problems for students. Here a common mistake was to assume that a geodesic which is initially radial will remain so – this true for the spacetime in question (which is spherically symmetric), but required some justification (it is not true, for example, in the Kerr spacetime). Some students attempted to solve the geodesic equations directly instead of making use of conserved quantities, and this inevitably led to difficulties. A few students also struggled with the integral that needed to be done in part (d) (i), obtaining logarithmic expressions instead of trigonometric ones.

**Summary**

Overall the quality of the answers was very high. The external examiners' considered this year's exam to be difficult, and yet the students scored very well – I am impressed, especially given the difficult circumstances this year. The spread of questions attempted was as expected, and (also as expected) the more mathematically minded students who attempted question 2 typically did very well. In general, *FHS* students did slightly better than *MSc* students.

**C7.6: General Relativity II**

- Question 1: This question was attempted by all students. Part a) was solved correctly by the majority of students, as was part b) i), although a few students did not notice here that the result from a) iii) could be used. Part b) ii) was clearly the most difficult question, very few students scored full marks here. Part b) iii) was again easier and nearly everyone scored at least some points here, but at the same time nearly everyone struggled with finding a spacelike geodesic.
- Question 2: This question was the least popular, it was attempted only by a handful of students. Part a) was generally carried out very well, but several students struggled with the concept of gauge in general relativity in part b). Part c) was again correctly solved by almost all students who attempted it. The last part of question 2 was the most difficult one and here only a small number of students presented a good solution.
- Question 3: This question was attempted by nearly everyone. Drawing the Penrose diagrams did not pose any difficulties for the majority of students, however a few students struggled with defining the concept of a black hole. Part b) was also executed well, but a few students did not show that the integral curves of the normal vector field to a null hypersurface are null geodesics. Nearly every student had the right idea for solving c) i), although there were a few computational errors. The determination of the causal character of the hypersurfaces $\{r = const\}$ was again done correctly by nearly everyone, but many students struggled with the computation of the surface gravity. Most

students gained some points for c) iii) & iv), but hardly anyone gave full solutions here.

**C7.4: Introduction to Quantum Information**

**Question 1**

Students did very well on it and received a high average mark. Some struggled with part (e) and confused it with Simon's problem. Single marks were also lost in parts (a) e.g. some students viewed the Hadamard transformation as a preparation of an equal superposition of all states in the Hilbert space or of entangled states and part (b) when no justification to the minimum number of calls was provided.

**Question 2**

This was the most popular question. Students knew their bookwork pretty well. Most of them found part (f) difficult for it required thinking and explaining rather than manipulating equations.

**Question 3**

This was the least popular question. The bookwork parts (a-c) presented no difficulties and students did very well on them. The most challenging were parts (d), (e) and to some extend (f). In part (d) some students struggled with the probabilistic nature of the algorithm and the estimates of the success probability. In part (e) only few managed to use the spectral decomposition to derive the required formula in few basic steps. Some students, managed part (f) without completing part (e).

**C3.5: Lie Groups**

**Question 1**

This question was about the noncompact symplectic group $Sp(2n, R)$. It was the most elementary question in terms of the material covered, and all 8 candidates attempted it, with the majority getting marks in the 18-25 range. Most candidates got through the proof that the symplectic group was in fact a Lie group, though some were careless about quoting the appropriate theorems to justify this. The parts of the question aimed at understanding the algebraic structure of the group were generally done well, though only a couple of candidates really gave a good explanation of the geometric interpretation of the isomorphism in the $n = 1$ case.

**Question 2**

This question, on representation theory, proved less popular. The standard parts of the questions were done well, but putting everything together to show nonexistence of nontrivial finite-dimensional unitary representations of $SL(2, R)$ proved more challenging.

**Question 3**

This question was on Weyl groups and maximal tori with candidates attaining marks in the 18-25 range. Candidates generally had a good grasp of the maximal torus for $SO(2n + 1)$ and how to find Weyl group elements. The more subtle case of $SO(2n)$ was of course harder, but some people understood this well.

### C6.1: Numerical Linear Algebra

This was generally a successful exam with a range of marks including many high ones. There were a surprisingly large number of candidates who made significant attempts at all three questions rather than concentrating on two as required. Possibly because of the open book format, irrelevant bookwork material that was not asked was described by some.

The first question on the Singular Value Decomposition was attempted by almost all candidates and was generally done reasonably well. Some candidates were a little sloppy with their arguments, in particular in part (c) even if they identified a correct Polar Decomposition which some did not. A surprisingly large number believed that the sum of the singular values was equal to the matrix trace.

The second question on Chebyshev polynomial iterative methods was also popular and reasonably well done. The final part was again found challenging by many who either did not attempt it or, more usually, did not make any headway with it.

The third question on GMRES was attempted by just under half of the candidates and also attracted a range of marks: most attempts seem to have been purposeful and not just rushed in the last few minutes of the exam. The first four parts were generally well done, but the final two unseen parts either attracted full or zero marks in general. There were very few clean answers to the final part.

**C5.5: Perturbation Methods**

*Q1*

While this question was not popular, it was well done in general by those making a serious attempt. Weaker scripts did not show that the second integral for $J(x)$ was ord$(1/[x\epsilon])$. In addition, many candidates were unable to demonstrate that the higher order terms from the power series expansion of the exponential were consistent with the stated error bound.

*Q2*

This was a very popular question. In the first part, weaker scripts often did not give correct reasoning for the error estimate though essentially all candidates noted the importance of this result in later parts of the question. Accurately considering the expansions with respect to the intermediate variable in the later parts was frequently the most troublesome aspect of the question for candidates.

*Q3*

Again, a popular question. The first part on multiple scales was very well done in general. The second part, involving WKBJ expansions, was tackled well in the earlier stages though mistakes generally emerged as the calculation proceeded.

## E. Comments on performance of identifiable individuals

*Removed from public version*

## F. Names of members of the Board of Examiners

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Prof Lionel Mason (Mathematical Institute, University Of Oxford)
Dr Dmitri Pushkin (Department of Mathematics, University of York)
Prof Alex Schekochihin (Department of Physics, University of Oxford; Chair)
Dr David Skinner (Department of Applied Mathematics and Theoretical Physics, University of Cambridge)
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Prof Andreas Muench
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Prof Sid Parameswaran
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