A15089W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

KINETIC THEORY

Hilary Term 2022

THURSDAY, 13th JANUARY 2022, 09:30 am to 12:30 pm

You should submit answers to all three questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. The distribution function $f(\mathbf{x}, \mathbf{v}, t)$ for a dilute monatomic gas evolves according to the Boltzmann equation

$$\partial_t f + \mathbf{v} \cdot \nabla f = \mathcal{C}[f, f],$$

where $\mathcal{C}[f, f]$ is Boltzmann's binary collision operator.

(a) [4 marks] Show that Boltzmann's H-function $H = \int f \log f \, \mathrm{d}\mathbf{v}$ evolves according to

$$\partial_t H + \nabla \cdot \mathbf{J} = S,$$

where $\mathbf{J} = \int \mathbf{v} f \log f \, \mathrm{d}\mathbf{v}$, and show that $S \leqslant 0$.
[*You may assume that, for any function* $\psi(\mathbf{v})$,

$$\int \mathcal{C}[f, f] \, \psi(\mathbf{v}) \, \mathrm{d}\mathbf{v} = \frac{1}{4} \int \mathrm{d}\mathbf{v} \int \mathrm{d}\mathbf{v}_\star \int \mathrm{d}\theta \, \mathrm{d}\phi \; B(V, \theta) \left(f' f'_\star - f f_\star \right) \left(\psi + \psi_\star - \psi' - \psi'_\star \right),$$

where the symbols have their usual meanings.]

The gas occupies the half-space above a flat planar boundary with normal $\mathbf{n}$ pointing *into* the gas. For the rest of this question, the point $\mathbf{x}$ lies on the boundary, and the arguments $\mathbf{x}$ and t of f are omitted. For velocities $\mathbf{v}$ directed *away* from the boundary, $f(\mathbf{v})$ is given by

$$|\mathbf{v} \cdot \mathbf{n}| \, f(\mathbf{v}) = \int_{\mathbf{v}' \cdot \mathbf{n} < 0} R(\mathbf{v}', \mathbf{v}) \, |\mathbf{v}' \cdot \mathbf{n}| \, f(\mathbf{v}') \, \mathrm{d}\mathbf{v}'. \qquad (\star)$$

The function $R(\mathbf{v}', \mathbf{v})$ satisfies the two properties

$$R(\mathbf{v}', \mathbf{v}) \geqslant 0, \qquad \text{(i)}$$

$$\int_{\mathbf{v} \cdot \mathbf{n} > 0} R(\mathbf{v}', \mathbf{v}) \, \mathrm{d}\mathbf{v} = 1 \text{ for each } \mathbf{v}' \text{ with } \mathbf{v}' \cdot \mathbf{n} < 0. \qquad \text{(ii)}$$

(b) [5 marks] Give a qualitative interpretation of the properties (i) and (ii), and show that the mass flux of gas through the boundary is zero.

(c) [9 marks] Show that, for any convex function C,

$$\int \mathbf{v} \cdot \mathbf{n} \, f^{(0)}(\mathbf{v}) \, C\left(\frac{f(\mathbf{v})}{f^{(0)}(\mathbf{v})} \right) \mathrm{d}\mathbf{v} \leqslant 0,$$

where $f(\mathbf{v})$ for $\mathbf{v} \cdot \mathbf{n} > 0$ is given in terms of $f(\mathbf{v}')$ for $\mathbf{v}' \cdot \mathbf{n} < 0$ by $(\star)$.

(d) [7 marks] By choosing a suitable convex function C, show that

$$\mathbf{J} \cdot \mathbf{n} \leqslant -\frac{1}{\Theta} \left(\mathbf{q} + \mathbf{u} \cdot \mathsf{P} \right)$$

on the boundary, where $\mathbf{q}$, $\mathbf{u}$ and P are the heat flux, fluid velocity, and pressure tensor. Give definitions of these quantities in terms of f.

[*You may assume that, for any functions g and w, convex function C, and subset Ω of* $\mathbb{R}^3$,

$$C\left(\int_\Omega w(\mathbf{v}') g(\mathbf{v}') \, \mathrm{d}\mathbf{v}' \right) \leqslant \int_\Omega w(\mathbf{v}') C(g(\mathbf{v}')) \, \mathrm{d}\mathbf{v}',$$

when the function w is normalised such that

$$\int_\Omega w(\mathbf{v}') \, \mathrm{d}\mathbf{v}' = 1.$$

A function $C(x)$ is convex if its second derivative $C''(x) > 0$ for all x.]

2. Consider a plasma consisting of Maxwellian electrons (mass m_e, charge $q_e = -e$, density n_e, temperature T_e, thermal speed $v_{\rm the} = \sqrt{2T_e/m_e}$) and a cold ion beam (mass m_i, charge $q_i = Ze$, density n_i, velocity $u_i \gg v_{\rm the}$—much greater than the width of the electron distribution). The corresponding distribution functions, which can be considered to be one-dimensional (in the direction of the beam), are

$$F_e(v) = \frac{n_e}{\sqrt{\pi}\, v_{\rm the}}\, e^{-v^2/v_{\rm the}^2}, \qquad F_i(v) = n_i\delta(v - u_i). \tag{1}$$

(a) [4 marks] The dispersion relation that determines the complex increment p of linear perturbations in an electrostatic plasma is

$$\epsilon(p,k) = 1 - \sum_\alpha \frac{\omega_{\rm p\alpha}^2}{k^2}\frac{1}{n_\alpha}\int {\rm d}v\, \frac{1}{v - ip/k}\frac{\partial F_\alpha}{\partial v} = 0, \tag{2}$$

where $\omega_{\rm p\alpha} = (4\pi q_\alpha^2 n_\alpha/m_\alpha)^{1/2}$ is the plasma frequency of species α ($= e, i$), k is the wavenumber of the perturbation, and the velocity integral is along the Landau contour. Anticipating that an instability will be possible for $p/k \gg v_{\rm the}$, show that, in this limit, the dispersion relation is

$$1 + \frac{\omega_{\rm pe}^2}{p^2} - \frac{\omega_{{\rm p}i}^2}{(ku_i - ip)^2} = 0. \tag{3}$$

(b) [6 marks] Show that there is an instability that attains its fastest growth at $k = \omega_{\rm pe}/u_i$ and produces, at that wavenumber, waves with the real frequency and growth rate

$$\omega_k = \left[1 - \frac{1}{2}\left(\frac{Zm_e}{2m_i}\right)^{1/3}\right]\omega_{\rm pe}, \qquad \gamma_k = \frac{\sqrt{3}}{2}\left(\frac{Zm_e}{2m_i}\right)^{1/3}\omega_{\rm pe}. \tag{4}$$

(c) [8 marks] The equations of quasilinear theory for the distribution functions and the wave spectrum are

$$\frac{\partial F_\alpha}{\partial t} = \frac{\partial}{\partial v}\left[\frac{q_\alpha^2}{m_\alpha^2}\sum_k \frac{\gamma_k|E_k|^2}{(kv - \omega_k)^2 + \gamma_k^2}\right]\frac{\partial F_\alpha}{\partial v}, \qquad \frac{\partial |E_k|^2}{\partial t} = 2\gamma_k|E_k|^2, \tag{5}$$

where E_k is the electric-field amplitude at wavenumber k, ω_k is the real frequency of the waves, and γ_k is their growth rate at that wavenumber. Assuming that, at least in the initial stage of the quasilinear evolution, the distribution functions will preserve their functional form (1), but T_e and u_i will change slowly with time, show that the kinetic-energy densities of the electrons and ions will satisfy

$$\frac{\rm d}{{\rm d}t}\int {\rm d}v\, \frac{m_e v^2}{2}F_e = \frac{{\rm d}\mathscr{E}}{{\rm d}t}, \qquad \frac{\rm d}{{\rm d}t}\frac{m_i n_i u_i^2}{2} = -2\frac{{\rm d}\mathscr{E}}{{\rm d}t}, \tag{6}$$

where $\mathscr{E} = \sum_k |E_k|^2/8\pi$ is the energy density of the electric field. You may assume that wavenumber sums are dominated by the fastest-growing mode described by (4).

(d) [3 marks] Based on (6), describe the energy flows in the system and, qualitatively, the evolution of the distribution functions.

(e) [4 marks] Make an educated qualitative guess about the typical size of $\mathscr{E}$ at which the instability saturates, i.e., the distribution becomes stable.

3. We consider a $3D$ stellar system composed of N particles of individual mass m, coupled through the pairwise gravitational interaction $U(\mathbf{x},\mathbf{x}') = -G/|\mathbf{x}-\mathbf{x}'|$.

(a) [4 marks] The system's instantaneous state is described by the distribution function (DF)

$$F_{\mathrm{d}}(\mathbf{x},\mathbf{v},t) = \sum_{i=1}^{N} m\, \delta_{\mathrm{D}}(\mathbf{x}-\mathbf{x}_i(t))\, \delta_{\mathrm{D}}(\mathbf{v}-\mathbf{v}_i(t)). \tag{7}$$

Show that F_{d} evolves according to

$$\frac{\partial F_{\mathrm{d}}}{\partial t} + \big[F_{\mathrm{d}}, H_{\mathrm{d}}\big] = 0, \tag{8}$$

and give expressions for the Hamiltonian H_{d} and the operator $[\,\cdot\,,\,\cdot\,]$.

(b) [4 marks] The system's mean-field state is described by $F_0 = \langle F_{\mathrm{d}}\rangle$ and $H_0 = \langle H_{\mathrm{d}}\rangle$. State the meaning of the symbol $\langle\,\cdot\,\rangle$. Assume that the system is in an integrable equilibrium associated with some angle-action coordinates $(\boldsymbol{\theta},\mathbf{J})$. Give two properties of angle-action coordinates. What can you say about F_0 and H_0 in angle-action coordinates?

(c) [5 marks] Instantaneous perturbations in the system's DF and potential are respectively denoted as $\delta F(\mathbf{x},\mathbf{v},t)$ and $\delta\Phi(\mathbf{x},t)$. Express $\delta\Phi$ as a function of δF. Write down the equation giving the time evolution of δF at first order in the perturbations, and write it down explicitly in angle-action coordinates. Explain briefly the physical meaning of each term.

(d) [4 marks] The Laplace–Fourier transform of any $F(\boldsymbol{\theta},\mathbf{J},t)$ is defined to be

$$\widetilde{F}_{\mathbf{k}}(\mathbf{J},\omega) = \int_0^{+\infty}\!\mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t}\!\int\!\frac{\mathrm{d}\boldsymbol{\theta}}{(2\pi)^3}\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\boldsymbol{\theta}}\, F(\boldsymbol{\theta},\mathbf{J},t), \tag{9}$$

with $\mathbf{k}\in\mathbb{Z}^3$. Stating clearly all your assumptions, show that the time evolution equation for δF can be rewritten as

$$\delta\widetilde{F}_{\mathbf{k}}(\mathbf{J},\omega) = -\frac{\mathbf{k}\cdot\partial F_0/\partial\mathbf{J}}{\omega-\mathbf{k}\cdot\boldsymbol{\Omega}(\mathbf{J})}\,\delta\widetilde{\Phi}_{\mathbf{k}}(\mathbf{J},\omega) - \frac{\delta F_{\mathbf{k}}(\mathbf{J},0)}{\mathrm{i}(\omega-\mathbf{k}\cdot\boldsymbol{\Omega}(\mathbf{J}))}, \tag{10}$$

where the definition of $\boldsymbol{\Omega}(\mathbf{J})$ has to be given.

(e) [4 marks] In the long term, the perturbations present in the system drive the relaxation of the system's overall orbital structure. Show that it is governed by

$$\frac{\partial F_0(\mathbf{J},t)}{\partial t} = -\frac{\partial}{\partial\mathbf{J}}\cdot\left[\int\frac{\mathrm{d}\boldsymbol{\theta}}{(2\pi)^3}\left\langle\frac{\partial\delta F}{\partial\boldsymbol{\theta}}\,\delta\Phi\right\rangle\right]. \tag{11}$$

(f) [4 marks] Equation (11) can be brought to the form

$$\frac{\partial F_0(\mathbf{J},t)}{\partial t} = -\frac{\partial}{\partial\mathbf{J}}\cdot\left[\mathbf{D}_1(\mathbf{J})\,F_0(\mathbf{J},t) - \mathbf{D}_2(\mathbf{J})\cdot\frac{\partial F_0}{\partial\mathbf{J}}\right], \tag{12}$$

for some $\mathbf{D}_1(\mathbf{J})$ and $\mathbf{D}_2(\mathbf{J})$. Assuming that F_0 is in thermal equilibrium at temperature $T=(k_{\mathrm{B}}\beta)^{-1}$, show that

$$\mathbf{D}_1(\mathbf{J}) = \mathbf{D}_2(\mathbf{J})\cdot\mathbf{K}(\mathbf{J}), \tag{13}$$

where the expression of the vector $\mathbf{K}(\mathbf{J})$ should be identified.