A15089W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

KINETIC THEORY

Hilary Term 2020

THURSDAY, 16TH JANUARY 09:30 am to 12:30 pm

You should submit answers to all three questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. The Boltzmann equation for a distribution of particles with unit mass and unit charge in an electric field $\mathbf{E}$ and magnetic field $\mathbf{B}$ with the Bhatnagar–Gross–Krook collision operator is

$$\frac{\partial f}{\partial t} + \mathbf{v}\cdot\nabla f + (\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\nabla_{\mathbf{v}} f = -\frac{1}{\tau}\left(f - f^{(0)}\right), \text{ where } f^{(0)} = \frac{\rho}{(2\pi\theta)^{3/2}}\exp\left(-\frac{|\mathbf{v}-\mathbf{u}|^2}{2\theta}\right),$$

and τ is a positive constant. The electromagnetic field is written in units for which the speed of light does not appear in this equation. You may assume that the electromagnetic field is prescribed, and unchanged by the particles.

(a) [3 marks] Give physical interpretations of the quantities ρ, $\mathbf{u}$, θ, and explain how they are calculated from f.

(b) [4 marks] Show that the quantity $H = \int f \log f \,\mathrm{d}\mathbf{v}$ satisfies an equation of the form

$$\frac{\partial H}{\partial t} + \nabla\cdot\mathbf{J} = S.$$

Give expressions for $\mathbf{J}$ and S in terms of f, and show that $S \leqslant 0$.

(c) [8 marks] Derive evolution equations for ρ and $\rho\mathbf{u}$, and show that the momentum flux $\mathbf{\Pi}$ obeys an equation of the form

$$\frac{\partial}{\partial t}\Pi_{ij} + \frac{\partial}{\partial x_k}Q_{ijk} - \rho(u_iE_j + u_jE_i) + \Pi_{ik}\epsilon_{kjl}B_l + \Pi_{jk}\epsilon_{kil}B_l = -\frac{1}{\tau}\left(\Pi_{ij} - \Pi_{ij}^{(0)}\right). \qquad (\dagger)$$

Give an expression for the tensor Q in terms of f.

(d) [10 marks] Now suppose that the electric field is zero, the magnetic field $\mathbf{B} = B\hat{\mathbf{k}}$ is constant and in the z-direction, and the distribution function f is spatially uniform.

Show that ($\dagger$) has solutions of the form $\Pi_{ij} = p\delta_{ij} + T_{ij}$ with $p = \rho\theta$ constant and $\mathbf{u} = \mathbf{0}$, provided T satisfies

$$\frac{\partial \mathsf{T}}{\partial t} + B\begin{pmatrix} -2T_{xy} & T_{xx}-T_{yy} & 0 \\ T_{xx}-T_{yy} & 2T_{xy} & 0 \\ 0 & 0 & 0 \end{pmatrix} = -\frac{1}{\tau}\mathsf{T}.$$

Hence show that T can undergo decaying oscillations, and find their frequency.

Why are these solutions with $\mathbf{u} = \mathbf{0}$ consistent with the momentum evolution equation?

2. Consider a plasma consisting of electrons (mass m_e, charge $q_e = -e$, density n_e, temperature T_e) and ions (mass m_i, charge $q_i = Ze$, density n_i, temperature T_i) in a one-dimensional, spatially homogeneous, constant in time, static (no mean flows) Maxwellian equilibrium. Allow infinitesimal perturbations of both electron and ion distribution functions, $f_\alpha = f_{0\alpha} + \delta f_\alpha$, where $\alpha = i, e$, and where $f_{0\alpha} = n_\alpha e^{-v^2/v_{\mathrm{th}\alpha}^2}/\sqrt{\pi}v_{\mathrm{th}\alpha}$, and $v_{\mathrm{th}\alpha} = \sqrt{2T_\alpha/m_\alpha}$ (n_α and T_α are both constant in space and time). Allow also infinitesimal electric perturbations $E = -\partial\varphi/\partial x$ and no magnetic fields, either in equilibrium or perturbed.

   (a) [5 marks] Starting from linearised, collisionless kinetic equation for the perturbed ion distribution function, show that the perturbed ion density $\delta n_i = \int \mathrm{d}v\, \delta f_i$ and flow velocity $u_i = (1/n_i)\int \mathrm{d}v\, v\delta f_i$ satisfy the following "fluid" equations

$$\frac{\partial \delta n_i}{\partial t} + n_i \frac{\partial u_i}{\partial x} = 0, \tag{1}$$

$$m_i n_i \frac{\partial u_i}{\partial t} + \frac{\partial \delta p_i}{\partial x} - Zen_i E = 0, \tag{2}$$

   where $\delta p_i = \int \mathrm{d}v\, m_i v^2 \delta f_i$ is the perturbed ion pressure. Assuming the ions are cold ($T_i \ll T_e$), you may henceforth neglect the contribution of the ion pressure to the momentum equation.

   (b) [5 marks] Consider perturbations with frequencies ω and wave numbers k such that $\omega \ll kv_{\mathrm{the}}$. Argue therefore that the linearised, collisionless kinetic equation for the perturbed electron distribution function can be approximated by

$$v\frac{\partial \delta f_e}{\partial x} - \frac{e}{m_e} E \frac{\partial f_{0e}}{\partial v} = 0. \tag{3}$$

   Hence show that the sole restoring force acting on the ion-flow perturbation is provided by the electron pressure gradient. Why are electron and ion dynamics different?

   (c) [5 marks] Solve (3) and show also that electrons are isothermal, viz., that their perturbed pressure is related to their perturbed density via $\delta p_e = T_e \delta n_e$. Why is this different from what happens in Langmuir waves?

   (d) [5 marks] Using Poisson's law and assuming $k\lambda_{\mathrm{De}} \ll 1$, where λ_{De} is the electron Debye length, relate δn_e to δn_i and show that the ion fluid equations reduce to

$$\frac{\partial^2 \delta n_i}{\partial t^2} = \frac{ZT_e}{m_i}\frac{\partial^2 \delta n_i}{\partial x^2} \tag{4}$$

   and the same equation for u_i. What kind of waves are described by this equation?

   (e) [5 marks] Without derivation, explain by what mechanism(s) these waves will be damped in a collisionless plasma and why this damping has not been captured by the above derivation? Which terms that we neglected contained the damping effects? Under what physical conditions would sound waves be heavily damped on the ions?

3. We consider a $3D$ stellar system composed of N particles of individual mass m, coupled through the pairwise gravitational interaction $U(\mathbf{x}, \mathbf{x}') = -G/|\mathbf{x} - \mathbf{x}'|$.

(a) [3 marks] The system's instantaneous state is described by the distribution function (DF)

$$F_{\mathrm{d}}(\mathbf{x}, \mathbf{v}, t) = \sum_{i=1}^{N} m\, \delta_{\mathrm{D}}(\mathbf{x} - \mathbf{x}_i(t))\, \delta_{\mathrm{D}}(\mathbf{v} - \mathbf{v}_i(t)), \tag{5}$$

and the associated instantaneous potential $\Phi_{\mathrm{d}}(\mathbf{x}, t)$. Write down, without derivation, the two coupled equations that F_{d} and Φ_{d} must satisfy.

(b) [3 marks] The system's mean-field state is given by $F_0 = \langle F_{\mathrm{d}} \rangle$, and $H_0 = |\mathbf{v}|^2/2 + \Phi_0(\mathbf{x})$, with $\Phi_0 = \langle \Phi_{\mathrm{d}} \rangle$. State the meaning of the symbol $\langle \cdot \rangle$. Assume that the system is in an integrable equilibrium, associated with some angle-action coordinates $(\boldsymbol{\theta}, \mathbf{J})$. Give two properties of angle-action coordinates. What can you say about F_0 and H_0 in angle-action coordinates?

(c) [5 marks] Instantaneous perturbations in the system's DF and potential are respectively denoted as $\delta F(\mathbf{x}, \mathbf{v}, t)$ and $\delta \Phi(\mathbf{x}, t)$. Write down the equation giving the time evolution of δF at first order in the perturbations, and write it down explicitly in angle-action coordinates. Explain briefly the physical meaning of each term.

(d) [3 marks] The Laplace-Fourier transform is defined to be

$$\delta \widetilde{F}_{\mathbf{k}}(\mathbf{J}, \omega) = \int_0^{+\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\omega t} \int \frac{\mathrm{d}\boldsymbol{\theta}}{(2\pi)^3}\, \mathrm{e}^{-\mathrm{i}\mathbf{k}\cdot\boldsymbol{\theta}}\, \delta F(\boldsymbol{\theta}, \mathbf{J}, t), \tag{6}$$

with $\mathbf{k} \in \mathbb{Z}^3$. Stating clearly all your assumptions, show that the time evolution equation for δF can be rewritten as

$$\delta \widetilde{F}_{\mathbf{k}}(\mathbf{J}, \omega) = -\frac{\mathbf{k} \cdot \partial F_0/\partial \mathbf{J}}{\omega - \mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J})}\, \delta \widetilde{\Phi}_{\mathbf{k}}(\mathbf{J}, \omega) - \frac{\delta F_{\mathbf{k}}(\mathbf{J}, 0)}{\mathrm{i}(\omega - \mathbf{k} \cdot \boldsymbol{\Omega}(\mathbf{J}))}, \tag{7}$$

where you should provide the definition of $\boldsymbol{\Omega}(\mathbf{J})$.

(e) [5 marks] In the long-term, the perturbations present in the system drive a long-term evolution of the system's overall orbital structure. Show that it is governed by

$$\frac{\partial F_0(\mathbf{J}, t)}{\partial t} = -\frac{\partial}{\partial \mathbf{J}} \cdot \left[\int \frac{\mathrm{d}\boldsymbol{\theta}}{(2\pi)^3} \left\langle \frac{\partial \delta F}{\partial \boldsymbol{\theta}}\, \delta \Phi \right\rangle \right]. \tag{8}$$

(f) [6 marks] Assume that a population of *massless* tracer particles, described by the mean-field DF, $G_0(\mathbf{J}, t)$, also orbits within the stellar system. Stating clearly all your assumptions, show that the tracer population undergoes an orbital relaxation described by

$$\frac{\partial G_0(\mathbf{J}, t)}{\partial t} = \frac{\partial}{\partial \mathbf{J}} \cdot \left[\mathbf{D}_2(\mathbf{J}) \cdot \frac{\partial G_0}{\partial \mathbf{J}} \right], \tag{9}$$

where the diffusion tensor $\mathbf{D}_2(\mathbf{J})$ is independent of $G_0(\mathbf{J}, t)$. [An explicit expression for $\mathbf{D}_2(\mathbf{J})$ is **not** required.] Discuss the particular form of equation (9).