Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

KINETIC THEORY

Hilary Term 2019

THURSDAY, 10TH JANUARY 2019, 09:30 am to 12:30 pm

You should submit answers to all three questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. The Boltzmann equation for a function $f(\mathbf{x}, \mathbf{v}, t)$ with the BGK collision operator is

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f = C[f], \text{ where } C[f] = -\frac{1}{\tau}\left(f - f^{(0)}\right), \tag{$\star$}$$

and τ is a positive constant. The particles may be assumed to have unit mass, and

$$f^{(0)} = \frac{\rho}{(2\pi\theta)^{3/2}} \exp\left(-\frac{|\mathbf{v}-\mathbf{u}|^2}{2\theta}\right).$$

(a) [4 marks] Give physical interpretations of the quantities ρ, $\mathbf{u}$, θ, and show that they are conserved by the BGK collision operator.

(b) [4 marks] Show that, for all distribution functions f, the BGK collision operator satisfies

$$\int \mathrm{d}\mathbf{v}\,(\log f)\, C[f] \leqslant 0.$$

Under what conditions does equality hold?

Now suppose that $\mathbf{x} = (x, y, z)$, $\mathbf{u} = (u, 0, 0)$, $\mathbf{v} = (v, \mathbf{v}_\perp)$ with $\mathbf{v}_\perp = (v_y, v_z)$, and that f is independent of y and z.

(c) [8 marks] If f satisfies the equation ($\star$), show that the functions

$$g(x, v, t) = \int \mathrm{d}\mathbf{v}_\perp\, f(x, \mathbf{v}, t), \quad h(x, v, t) = \int \mathrm{d}\mathbf{v}_\perp\, |\mathbf{v}_\perp^2|\, f(x, \mathbf{v}, t),$$

satisfy a closed system of the form

$$\frac{\partial g}{\partial t} + v\frac{\partial g}{\partial x} = -\frac{1}{\tau}\left(g - g^{(0)}\right), \quad \frac{\partial h}{\partial t} + v\frac{\partial h}{\partial x} = -\frac{1}{\tau}\left(h - h^{(0)}\right), \tag{$\dagger$}$$

and determine the functions $g^{(0)}$ and $h^{(0)}$. Give also expressions for the quantities ρ, u, θ in this system.

(d) [9 marks] Starting from the system ($\dagger$) derive evolution equations for the macroscopic mass, momentum, and energy densities. Show further that the momentum and energy fluxes can be written as functions of ρ, u, θ, and two further quantities:

$$T = -2\rho\theta + \int \mathrm{d}v\, h, \quad q = \frac{1}{2}\int \mathrm{d}v\, w^3 g + \frac{1}{2}\int \mathrm{d}v\, wh.$$

Give physical interpretations of these last two quantities, and verify that they vanish when $g = g^{(0)}$ and $h = h^{(0)}$.

2. Consider one-dimensional, electrostatic perturbations in a two-species (electron-ion) plasma whose equilibrium distribution function (with respect to velocities in the direction of the spatial variation of perturbations) is $F_\alpha(v)$, where α stands for electrons (e) or ions (i). The dielectric function of such a plasma is

$$\epsilon(p,k) = 1 - \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{k^2}\frac{1}{n_\alpha}\int \mathrm{d}v\, \frac{F_\alpha'(v)}{v - ip/k}, \tag{1}$$

where p is the Laplace variable, $\omega_{\mathrm{p}\alpha}$ is the plasma frequency of species α, n_α is the mean density of that species, k is the wave number of the perturbation, and $F_\alpha'(v) = \partial F_\alpha/\partial v$.

(a) [5 marks] Explain without derivation what will be the time evolution of an initial electric perturbation in a such a plasma and what role in that evolution will be played by the solutions of the equation $\epsilon(p,k) = 0$. Explain, also without derivation, how one must choose the integration contour in (1).

(b) [5 marks] Let the electron distribution function be a "double Lorentzian" consisting of two counterpropagating beams with velocity u_b and width Δ, viz.,

$$F_e(v) = \frac{n_e\Delta}{2\pi}\left[\frac{1}{(v-u_\mathrm{b})^2+\Delta^2} + \frac{1}{(v+u_\mathrm{b})^2+\Delta^2}\right], \tag{2}$$

while the ions are Maxwellian with thermal speed $v_{\mathrm{th}i} \ll u_\mathrm{b}$. Assume also that p/k will be of the same order as u_b and hence that the ion contribution to (1) is negligible. By integrating by parts and then choosing the integration contour judiciously, or otherwise, calculate the dielectric function $\epsilon(p,k)$ for this plasma and hence show that the dispersion relation is

$$\sigma^4 + (2u_\mathrm{b}^2 + v_\mathrm{p}^2)\sigma^2 + u_\mathrm{b}^2(u_\mathrm{b}^2 - v_\mathrm{p}^2) = 0, \tag{3}$$

where $\sigma = \Delta + p/k$ and $v_\mathrm{p} = \omega_{\mathrm{pe}}/k$.

(c) [5 marks] In the long-wavelength limit, viz., $k \ll \omega_{\mathrm{pe}}/u_\mathrm{b}$, find the condition for an instability to exist and calculate the growth rate of this instability. Is the nature of this instability kinetic (due to Landau resonance) or hydrodynamic?

(d) [5 marks] Consider the case of cold beams, $\Delta = 0$. Without making any *a priori* assumptions about k, calculate the maximum growth rate of the instability. Sketch the growth rate as a function of k.

(e) [5 marks] Allowing warm beams, $\Delta > 0$, show that the system is unstable provided

$$u_\mathrm{b} > \Delta \quad \text{and} \quad k < \omega_{\mathrm{pe}}\frac{\sqrt{u_\mathrm{b}^2 - \Delta^2}}{u_\mathrm{b}^2 + \Delta^2}. \tag{4}$$

Comment on the effect that a finite beam width has on the stability of the system and the kind of perturbations that can grow.

3. (a) [4 marks] A stellar system has distribution function (DF) $f(\mathbf{x}, \mathbf{v}, t)$. Explain what a 'mean-field' model of the system is, and why it is reasonable to assume its DF can be written $f_0(\mathbf{J})$ where the components of $\mathbf{J}$ are action integrals. Explain physically why over sufficiently long times the optimal mean-field model of any stellar system changes.

(b) [7 marks] Show that

$$\frac{\partial f_0}{\partial t} = -\frac{\partial}{\partial \mathbf{J}} \cdot \mathbf{F},$$

where

$$\mathbf{F} = \mathrm{i} \sum_{\mathbf{n}} \mathbf{n} \left\langle \hat{f}_1(\mathbf{n}, \mathbf{J}, t) \hat{\Phi}_1(-\mathbf{n}, \mathbf{J}, t) \right\rangle.$$

Here the sum is over vectors with integer components and the meaning of $\hat{f}_1$ and $\hat{\Phi}_1$ should be explained.

(c) [6 marks] Let

$$\widetilde{f}_1(\mathbf{n}, \mathbf{J}, p) \equiv \int_0^\infty \mathrm{d}t \, \mathrm{e}^{-pt} \hat{f}_1(\mathbf{n}, \mathbf{J}, t)$$

be the Laplace transform of $\hat{f}$ and similarly for $\widetilde{\Phi}_1$. By obtaining a relation between $\widetilde{f}_1$ and $\widetilde{\Phi}_1$, or otherwise, show that

$$\mathbf{F} = \mathrm{i} \left\langle \sum_{\mathbf{n}} \mathbf{n} \int \frac{\mathrm{d}p}{2\pi \mathrm{i}} \mathrm{e}^{pt} \left(\frac{\mathrm{i}\mathbf{n} \cdot \frac{\partial f_0}{\partial \mathbf{J}} \widetilde{\Phi}_1(\mathbf{n}, \mathbf{J}, p) + \hat{f}_1(\mathbf{n}, \mathbf{J}, 0)}{p + \mathrm{i}\mathbf{n} \cdot \mathbf{\Omega}} \right) \int \frac{\mathrm{d}p'}{2\pi \mathrm{i}} \mathrm{e}^{p't} \widetilde{\Phi}_1(-\mathbf{n}, \mathbf{J}, p') \right\rangle.$$

Write down an expression for the part $\mathbf{F}_2$ of $\mathbf{F}$ that depends on $\partial f_0 / \partial \mathbf{J}$ and explain the physical significance of both this part and the remainder of $\mathbf{F}$.

(d) [6 marks] Obtain an expression, valid to first order in the fluctuating potential $\Phi_1(\boldsymbol{\theta}, \mathbf{J}, t)$, for the change $\mathbf{J}_1(t)$ in a star's actions that is induced during time t. Show that the components $J_{1i}(t)$ of $\mathbf{J}_1$ satisfy

$$\left. \frac{\mathrm{d}}{\mathrm{d}t'} \right|_{t=t'} \langle J_{1i}(t) J_{1j}(t') \rangle = \sum_{\mathbf{n}} n_i n_j \left\langle \int \frac{\mathrm{d}p}{2\pi \mathrm{i}(p + \mathrm{i}\mathbf{n} \cdot \mathbf{\Omega})} \int \frac{\mathrm{d}p'}{2\pi \mathrm{i}} \mathrm{e}^{(p+p')t} \widetilde{\Phi}_1(\mathbf{n}, \mathbf{J}_0, p) \widetilde{\Phi}_1(-\mathbf{n}, \mathbf{J}_0, p') \right\rangle,$$

where $\mathbf{J}_0$ comprises the unperturbed actions.

(e) [2 marks] Comment on the similarity between this expression and your expression for $\mathbf{F}_2$.