Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

KINETIC THEORY

Hilary Term 2016

THURSDAY, 14th JANUARY 2016, 9:30am to 12:30pm

You should submit answers to all three questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. **Kinetics of gases.**

(a) A distribution of particles of unit mass evolves according to the Boltzmann equation

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \nabla f = C[f]. \tag{1}$$

(i) [3 marks] Show that $H = \int f \log f \, \mathrm{d}\mathbf{v}$ obeys the evolution equation

$$\frac{\partial H}{\partial t} + \nabla \cdot \mathbf{J} = S, \tag{2}$$

where

$$\mathbf{J} = \int (f \log f)\, \mathbf{v} \, \mathrm{d}\mathbf{v}, \quad S = \int (\log f)\, C[f] \, \mathrm{d}\mathbf{v}. \tag{3}$$

(ii) [6 marks] Suppose the gas is confined to a domain Ω by a boundary $\partial\Omega$ on which the particles undergo specular (mirror) reflections. Find an evolution equation for $\mathcal{H} = \int_\Omega H \, \mathrm{d}\mathbf{x}$.

[Hint: Consider a point on the boundary with normal $\mathbf{n}$*. How is* f *for particles that have just collided with the boundary related to* f *for particles that are about to collide with the boundary?]*

(iii) [4 marks] Now suppose that $C[f]$ is the BGK collision operator. Show that $\mathcal{H}$ is non-increasing, and find the condition(s) under which $\mathcal{H}$ is unchanging in time.

(b) Now suppose that f represents a distribution of marked tracer particles that collide only with a much more numerous distribution of unmarked particles through the BGK collision operator with timescale τ. The unmarked particles are distributed according to a Maxwell–Boltzmann distribution with prescribed temperature $\theta = 1/2$ and zero mean velocity. The number density n of tracer particles is conserved during collisions.

(i) [3 marks] Suppose in addition that f does not depend upon the y and z coordinates. Show that the reduced distribution $\overline{f} = \int f \, \mathrm{d}v_y \, \mathrm{d}v_z$ evolves according to

$$\frac{\partial \overline{f}}{\partial t} + v_x \frac{\partial \overline{f}}{\partial x} = -\frac{1}{\tau}\left(\overline{f} - \frac{n}{\sqrt{\pi}}\, \mathrm{e}^{-v_x^2}\right). \tag{4}$$

(ii) [9 marks] Use the first two terms in a multiple-scales expansion of $\overline{f}$ to find an evolution equation for n over timescales much longer than τ.

[Hint: What solvability condition(s) should you apply?]

2. **Kinetics of plasmas.** Because electrons have much smaller mass than the ions, their collision frequency in a plasma is typically much larger than that of the ions, $\nu_e \gg \nu_i$. In this limit, treating electrons as collisional and ions as collisionless, it is often possible to assume that the electrons have a Boltzmann distribution:

$$f_e = \frac{n_e}{(\pi v_{\rm the}^2)^{3/2}} \exp\left[-\frac{1}{T_e}\left(\frac{m_e v^2}{2} - e\varphi\right)\right] \approx f_{0e}\left(1 + \frac{e\varphi}{T_e}\right) = f_{0e}\left(1 + \Phi\right), \tag{5}$$

where n_e is the mean electron density, $v_{\rm the} = \sqrt{2T_e/m_e}$, m_e their mass, T_e their temperature (assumed constant), $-e$ their charge and φ the electric-potential perturbation, $\Phi \equiv e\varphi/T_e \ll 1$ its normalised version, assumed small, and f_{0e} is the equilibrium Maxwellian distribution. Thus, the perturbed electron distribution function is $\delta f_e = f_{0e}\Phi$.

(a) [5 marks] Consider a spatially homogeneous hydrogen plasma consisting of collisionless ions (protons) and Boltzmann electrons. Assume a Maxwellian equilibrium distribution of the ions, f_{0i}, with mean temperature T_i and density n_i, and consider infinitesimal perturbations δf_i of this distribution in response to infinitesimal electrostatic perturbations φ. Show that, in the limit $k\lambda_{\rm De} \ll 1$, where $\lambda_{\rm De}$ is the electron Debye length, such perturbations with wave number $\mathbf{k}$ satisfy

$$\frac{\partial \delta f_i}{\partial t} + i\mathbf{k}\cdot\mathbf{v}\,\delta f_i = i c_{\rm s}^2 \Phi\,\mathbf{k}\cdot\frac{\partial f_{0i}}{\partial \mathbf{v}}, \quad \Phi = \frac{1}{n_i}\int {\rm d}^3\mathbf{v}\,\delta f_i, \tag{6}$$

where $c_{\rm s} = \sqrt{T_e/m_i}$ and m_i the ion mass.

(b) [9 marks] By considering an initial-value problem for this system, derive the expression for the dielectric function and show that the dispersion relation for the modes of oscillation of the perturbations of this system, $\Phi \propto \exp(-i\omega t + \gamma t)$, is

$$1 + \frac{T_e}{T_i}\left[1 + \zeta Z(\zeta)\right] = 0, \quad \text{where} \quad Z(\zeta) = \frac{1}{\sqrt{\pi}}\int_{C_{\rm L}} {\rm d}u\,\frac{e^{-u^2}}{u - \zeta}, \tag{7}$$

$\zeta = (\omega + i\gamma)/kv_{\rm th\mathit{i}}$, $v_{\rm th\mathit{i}} = \sqrt{2T_i/m_i}$, and the integral in the definition of the plasma dispersion function Z is taken along the Landau contour $C_{\rm L}$ (you do not need to give the justification for using the Landau contour, simply assume it is known).

(c) [8 marks] The plasma dispersion function has the following limiting forms:

$$Z(\zeta) \approx i\sqrt{\pi} - 2\zeta + \ldots \text{ for } |\zeta| \ll 1 \quad \text{and} \quad Z(\zeta) \approx i\sqrt{\pi}\,e^{-\zeta^2} - \frac{1}{\zeta} - \frac{1}{2\zeta^3} - \frac{3}{4\zeta^5} + \ldots, \tag{8}$$

the latter for $|\zeta| \gg 1$, $|{\rm Re}\,\zeta| \gg |{\rm Im}\,\zeta|$. In which of these limits does the dispersion relation (7) have a solution? Find this solution, obtaining expressions for the frequency of oscillations ω and their damping rate γ. Give a physical interpretation of the solution that you have obtained.

(d) [3 marks] Show that your solution is valid only if the ions are sufficiently cold and derive the condition their temperature must satisfy. Explain physically why the Landau damping rate in this limit is much smaller than the frequency of the waves.

3. **Kinetics of self-gravitating systems.**

(a) [5 marks] Let $f_0(\mathbf{J})$ be the distribution function (DF) of an equilibrium stellar system that has gravitational potential $\Phi_0(\mathbf{x})$ and angle-action coordinates $(\boldsymbol{\theta}, \mathbf{J})$. Show that if we write the DF of the perturbed model $f(\mathbf{x}, \mathbf{v}, t) = f_0 + f_1(\mathbf{x}, \mathbf{v}, t)$, then to first order in the perturbations f_1 satisfies

$$\frac{\partial f_1}{\partial t} + \boldsymbol{\Omega}_0 \cdot \frac{\partial f_1}{\partial \boldsymbol{\theta}} - \frac{\partial f_0}{\partial \mathbf{J}} \cdot \frac{\partial \Phi_1}{\partial \boldsymbol{\theta}} = 0,$$

where $\boldsymbol{\Omega}_0 = \partial H_0 / \partial \mathbf{J}$ and the perturbed potential is $\Phi(\mathbf{x}, t) = \Phi_0(\mathbf{x}) + \Phi_1(\mathbf{x}, t)$. Hence or otherwise show that

$$\widetilde{f}_1(\mathbf{n}, \mathbf{J}, p) = \frac{\mathrm{i}\mathbf{n} \cdot \frac{\partial f_0}{\partial \mathbf{J}} \widetilde{\Phi}_1(\mathbf{n}, \mathbf{J}, p) + \hat{f}_1(\mathbf{n}, \mathbf{J}, 0)}{p + \mathrm{i}\mathbf{n} \cdot \boldsymbol{\Omega}_0}, \qquad (1)$$

where the meanings of a tilde and a hat should be explained.

(b) [5 marks] What physical principle is used to obtain from the last equation the expression

$$\widetilde{\Phi}_1(\mathbf{n}', \mathbf{J}', p) = -(2\pi)^3 \int \mathrm{d}^3\mathbf{J} \sum_{\mathbf{n}} E_{\mathbf{n}'\mathbf{n}}(\mathbf{J}', \mathbf{J}, p) \frac{\hat{f}_1(\mathbf{n}, \mathbf{J}, 0)}{p + \mathrm{i}\mathbf{n} \cdot \boldsymbol{\Omega}_0}, \qquad (2)$$

where $\mathbf{E}$ is the inverse of the "dielectric tensor"? Explain (without calculation) how from this equation we can obtain

$$\widetilde{f}_1(\mathbf{n}, \mathbf{J}, p) = -(2\pi)^3 \mathrm{i} \frac{\mathbf{n} \cdot \frac{\partial f_0}{\partial \mathbf{J}}}{p + \mathrm{i}\mathbf{n} \cdot \boldsymbol{\Omega}_0} \int \mathrm{d}^3\mathbf{J}' \sum_{\mathbf{n}'} E_{\mathbf{n}\mathbf{n}'}(\mathbf{J}, \mathbf{J}', p) \frac{\hat{f}_1(\mathbf{n}', \mathbf{J}', 0)}{p + \mathrm{i}\mathbf{n}' \cdot \boldsymbol{\Omega}_0'} + \frac{\hat{f}_1(\mathbf{n}, \mathbf{J}, 0)}{p + \mathrm{i}\mathbf{n} \cdot \boldsymbol{\Omega}_0}. \quad (3)$$

(c) Fluctuations in Φ drive a diffusive flux $\mathbf{F}$ of the mass-bearing stars through phase space. $\mathbf{F}$ is given by

$$\mathbf{F}(\mathbf{J}) = \mathrm{i} \left\langle \sum_{\mathbf{n}} \mathbf{n} \int \frac{\mathrm{d}p}{2\pi\mathrm{i}} \mathrm{e}^{pt} \widetilde{f}_1(\mathbf{n}, \mathbf{J}, p) \int \frac{\mathrm{d}p'}{2\pi\mathrm{i}} \mathrm{e}^{p't} \widetilde{\Phi}_1(-\mathbf{n}, \mathbf{J}, p') \right\rangle,$$

where $\langle\cdot\rangle$ indicates an ensemble average. A population of massless tracer particles orbits within the stellar system. Let $g_0(\mathbf{J})$ and $g_1(\mathbf{x}, \mathbf{v}, t)$ be the unperturbed and perturbed DFs of this population.

(i) [10 marks] Show that the phase-space flux $\mathbf{G}$ of the tracer population is given by an expression of the form

$$\mathbf{G} = -\mathbf{D}_2(\mathbf{J}) \cdot \frac{\partial g_0}{\partial \mathbf{J}}.$$

[*an expression for* $\mathbf{D}_2$ *is* **not** *required*]

(ii) [5 marks] Explain the physical significance of the form taken by $\mathbf{G}$.