Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

INTRODUCTION TO QUANTUM CONDENSED MATTER PHYSICS

Trinity Term 2016

MONDAY, 13 JUNE 2016, 14.30 to 16.00

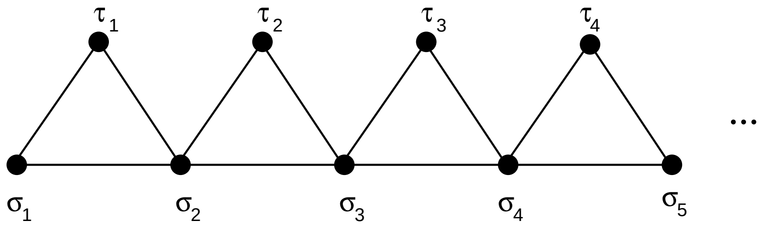
You should submit answers to two of the three questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. Consider an Ising model defined on the lattice shown below (the lattice has altogether $2L$ sites and we impose periodic boundary conditions)



The energy is given by

$$E = -J \sum_{j=1}^{L} \sigma_j \sigma_{j+1} + \sigma_j \tau_j + \tau_j \sigma_{j+1} \ , \tag{1}$$

where $J > 0$ and periodic boundary conditions impose $\sigma_{L+1} = \sigma_1$.

(a) [4 marks] Write down expressions for the partition function and the free energy per site at finite temperature. Give an expression for the average of the magnetization per site

$$\frac{1}{2L} \sum_j \sigma_j + \tau_j$$

at temperature $T > 0$.

(b) [6 marks] Describe the transfer matrix method for calculating the partition function for a system of spins $\sigma_1, \dots, \sigma_L$ on a ring (i.e. we impose periodic boundary conditions) described by an energy of the form $E = \sum_{j=1}^{L} E_0(\sigma_j, \sigma_{j+1})$ with $E_0(\sigma_j, \sigma_{j+1}) = E_0(\sigma_{j+1}, \sigma_j)$.

(c) [10 marks] Calculate the partition function for model defined in (1) at $T > 0$ by means of the transfer matrix method. What is the free energy per site in the thermodynamic limit?

(d) [5 marks] Calculate the thermal average of σ_1 in the limit of large L. Give a physical interpretation of your result.

2. Consider a one-dimensional quantum spin model with Hamiltonian

$$H = -\sum_{i=1}^{L} J^x S_i^x S_{i+1}^x + J^y S_i^y S_{i+1}^y + J^z S_i^z S_{i+1}^z \ , \tag{2}$$

where $J^x > J^y > J^z > 0$ and we have a spin-S on each site of the lattice, i.e. $(S_i^x)^2 + (S_i^y)^2 + (S_i^z)^2 = S(S+1)$. We impose periodic boundary conditions $S_{L+1}^\alpha = S_1^\alpha$.

(a) [5 marks] What are the classical ground states of this model? (Hint: consider (S_j^x, S_j^y, S_j^z) to be a classical vector of length S and assume translational invariance). What is the symmetry operation that relates the ground states?

(b) [3 marks] Explain the idea of spontaneous symmetry breaking for the example (2).

(c) [4 marks] Let $\tilde{S}_j^x$, $\tilde{S}_j^y$, $\tilde{S}_j^z$ be spin-S operators on site j of a one-dimensional lattice. Consider S to be large. The Holstein-Primakoff representation is defined by

$$\tilde{S}_j^z = S - a_j^\dagger a_j, \quad \tilde{S}_j^+ = \tilde{S}_j^x + i\tilde{S}_j^y = \sqrt{2S - a_j^\dagger a_j}\ a_j,\ [a_j, a_\ell^\dagger] = \delta_{j,\ell}.$$

Explain the nature and usefulness of this representation. Comment on complications that generally could arise.

(d) [6 marks] Apply the Holstein-Primakoff representation to the Hamiltonian (2). How should you choose the spin operators $\tilde{S}_j^\alpha$ to be related to S_j^β and why?

Carry out an expansion of H in inverse powers of S. Ignore the constant contribution and drop all terms that grow more slowly than S, when S becomes large. Show that the resulting Hamiltonian $H_{\rm LSW}$, the linear spin wave approximation to H, takes the form

$$H_{\rm LSW} = \sum_{j=1}^{L} A(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j) + B(a_j a_{j+1} + a_j^\dagger a_{j+1}^\dagger) + C a_j^\dagger a_j.$$

(e) [7 marks] Diagonalize the Hamiltonian $H_{\rm LSW}$ (you may drop constant contributions).

3. Consider the anharmonic oscillator

$$H(\lambda, \mu) = \frac{\hat{p}^2}{2m} + \frac{\kappa}{2}\hat{x}^2 + \frac{\lambda}{4!}\hat{x}^4 + \frac{\mu}{6!}\hat{x}^6.$$

Throughout this problem you may set $\hbar = 1$.

(a) [4 marks] What is the path integral representation for the partition function $Z(\beta)$? What kinds of paths are integrated over in the path integral?

(b) [3 marks] The imaginary-time Green's function of $H(0,0)$ is defined as

$$G^{(0)}(\tau_1 - \tau_2) = \langle T_\tau \bar{x}(\tau_1)\bar{x}(\tau_2)\rangle_\beta \ .$$

Explain what T_τ, $\bar{x}(\tau)$ and $\langle\rangle_\beta$ mean in this equation.

(c) [3 marks] What differential equation does $G^{(0)}(\tau)$ satisfy? What are the boundary conditions?

(d) [3 marks] The generating functional for $H(0,0)$ is defined as

$$W_0[J] \equiv \mathcal{N} \int \mathcal{D}x(\tau) \ \exp\left(-\frac{1}{2}\int_0^{\hbar\beta} d\tau \left[x(\tau)\hat{D}x(\tau) - 2J(\tau)x(\tau)\right]\right),$$

where

$$\hat{D} = -\frac{m}{\hbar}\frac{d^2}{d\tau^2} + \frac{\kappa}{\hbar}.$$

The generating functional can be expressed in the form

$$W_0[J] = W_0[0]\exp\left(\frac{1}{2}\int d\tau d\tau' J(\tau)G^{(0)}(\tau - \tau')J(\tau')\right).$$

Give the definition of the generating functional $W[J]$ for $H(\lambda, \mu)$ and argue that

$$W[J] = \exp\left(-\int_0^{\hbar\beta} d\tau' \left\{\frac{\lambda}{4!\hbar}\left[\frac{\delta}{\delta J(\tau')}\right]^4 + \frac{\mu}{6!\hbar}\left[\frac{\delta}{\delta J(\tau')}\right]^6\right\}\right) W_0[J].$$

(e) [7 marks] Express the first order in λ and μ corrections to the partition function in terms of the Green's function $G^{(0)}$. What are the corresponding Feynman diagrams?

(f) [5 marks] Draw the diagrams describing contributions to second order in λ and second order in μ respectively.