SECOND PUBLIC EXAMINATION

Master Course in Theoretical and Mathematical Physics

Radiative Processes and High-energy Astrophysics

TRINITY TERM 2022

Monday 13th June

Answer **both** *questions.*

Start the answer to each question in a fresh book.

A list of physical constants and conversion factors accompanies this paper.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do NOT turn over until told that you may do so.

1. (a) Neglecting scattering, the radiative transfer equation reads:

$$\frac{\mathrm{d}I_\nu}{\mathrm{d}s} = j_\nu - \alpha_\nu I_\nu \,.$$

Describe all elements of this equation, and find its formal solution. [4]

(b) Consider the case of a discrete atomic transition with energy $h\nu_{12}$, ignoring collisional excitations and de-excitations. Explain what the Einstein coefficients A_{21}, B_{21} and B_{12} are, and why the emission and absorption coefficients can be written as:

$$j_\nu = \frac{h\nu\, n_2}{4\pi} A_{21}\varphi(\nu)$$
$$\alpha_\nu = \frac{h\nu}{c}\left[n_1 B_{12} - n_2 B_{21}\right]\varphi(\nu)\,,$$

where n_1 and n_2 are the number densities of atoms in the lower and upper states respectively, and $\varphi(\nu)$ is the line profile. [5]

(c) Ignore spontaneous emission and assume an homogeneous medium of width ℓ. Using the solution of the radiative transfer equation found in (a), show that the emergent intensity is:

$$I_\nu = I_\nu^0 e^{-\tau_\nu}\,,$$

where I_ν^0 is the background intensity. What is τ_ν? How does it depend on the quantities above? The equivalent width of an absorption line is defined as:

$$W = \int \mathrm{d}\lambda\, \frac{I_\lambda^0 - I_\lambda}{I_\lambda^0}\,.$$

Show that, under certain approximations, W is given by:

$$W = \frac{h}{\nu_{12}} B_{12}\, N_1\,,$$

where $N_1 \equiv n_1\ell$ is the column density. What approximations did you use? [4]

(d) In the case of the Lyman-α forest ($h\nu_{12} = 10.2\,\mathrm{eV}$, $\nu_{12} = 2.47\times10^{15}\,\mathrm{Hz}$, $A_{21} = 6.3\times10^{8}\,\mathrm{Hz}$), typical measurements yield $W \sim 0.07\,\text{Å}$. Much of this forest is made of "cold" ($T_e \sim 10^4$ K) neutral hydrogen in cosmic filaments. Modelling them as $L \sim 10\,\mathrm{Mpc}$ long cylinders with diameter $d \sim 0.5\,\mathrm{Mpc}$, calculate the number density of neutral hydrogen n_{HI} in these filaments. [4]

(e) Assuming thermal equilibrium between photo-ionization and recombination, estimate the neutral fraction $1 - x_e$ of the gas hosted by these filaments (where x_e is the ionized fraction), given the binding energy of hydrogen is $\chi = 13.6\,\mathrm{eV}$. [4]

(f) The number you have obtained in (e) is a very poor under-estimate, as physical conditions lie far away from thermal equilibrium. In reality, the neutral fraction is never smaller than $1 - x_e \sim 10^{-6}$. Assuming this latter value of x_e, compute the Sunyaev-Zel'dovich Compton-y amplitude for the filaments. Given the level of uncertainty in current measurements, $\Delta y \sim 10^{-9}$, would these filaments be observable? [4]

2. (a) An ultra-relativistic electron is travelling through the interstellar medium (ISM). Explain what is meant by "ultra-relativistic". [1]

(b) Assume the electron has a Lorentz factor $\gamma = 10^6$. Is diffuse shock acceleration a viable mechanism to provide this electron with its energy? Write down the strong shock conditions that relate the upstream, downstream, and shock-front velocities in each of the three rest frames (upstream, downstream, and shock front), by defining the shock front velocity towards the upstream material as v_s in the upstream rest frame. Show how each time the electron crosses the shock front, it gains energy $\Delta E/E \propto V/c$, where V is the mean velocity of the particles on the other side of the shock front in the rest frame of the electron prior to crossing the shock front. [8]

(c) The electron encounters a region of the ISM permeated by a weak mean magnetic field $\vec{B}$ of strength $B = 10\mu G$. Describe the motion of the electron. From the Lorentz force acting on the electron, show that the gyro-frequency is $eB/(2\pi\gamma m)$, where e and m are the charge and mass of the electron. Compute the actual gyro-frequency of this electron in Hz. [4]

(d) Describe the beaming pattern from this electron seen by a distant observer. What is the opening angle between the nulls of the beam in the forward direction? Calculate the time duration of a radiation pulse, given your previous estimate of the gyro-frequency. How does this compare to the gyro-period? [8]

(e) A population of electrons with energies that follow a power law yields a broadband spectrum of synchrotron radiation. Observationally, this spectrum exhibits a pronounced decline at the high frequency end, often referred to as the high-frequency cut-off. Describe in detail two explanations for this high-frequency cut-off? [4]