Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

GROUPS AND REPRESENTATION

Hilary Term 2022

FRIDAY, 14th JANUARY 2022, 09:30 am to 12:30 pm

You should submit answers to ***three*** *out of the four questions.*

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The use of a calculator is **not** *allowed.*

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. (a) [6 marks] Show that the matrices

$$Q = \mathrm{diag}(\alpha, \alpha^*) \;, \qquad P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;,$$

where $\alpha = \exp(2\pi i/5)$, generate a group G of order 10 and find the conjugacy classes of this group. Show that the representation of G defined by the above matrices is irreducible.

(b) [5 marks] Show that, over complex vector spaces, G has two one-dimensional irreducible representations and two two-dimensional irreducible representations. Denote the trivial representation by R_0, the other one-dimensional representation by R_1 and the two-dimensional representations by R_2 and R_3, where R_2 is the representation given in part (a). Write down the character table of G.

(c) [5 marks] Find the Clebsch-Gordan decompositions (that is, the irreducible representation content) of $R_i \otimes R_j$, for all $i, j \in \{0, 1, 2, 3\}$.

(d) [5 marks] Consider a Yukawa term of the form $\lambda_{ab} H \bar{\psi}_L^a \psi_R^b$, where H is a complex scalar, ψ_L^a and ψ_R^b are left and right-handed fermions and $a, b = 1, 2, 3$. Fix the G-representation assignments $H \sim R_0$, $\bar{\psi}_L^3 \sim R_1$ and $\psi_R^3 \sim R_1$ and consider two choices for the assignments of the other fields: (i) $(\bar{\psi}_L^1, \bar{\psi}_L^2) \sim R_2$, $(\psi_R^1, \psi_R^2) \sim R_2$ and (ii) $(\bar{\psi}_L^1, \bar{\psi}_L^2) \sim R_2$, $(\psi_R^1, \psi_R^2) \sim R_3$. Determine the most general form of the Yukawa couplings λ_{ab} consistent with G invariance for case (i) and (ii).

(e) [4 marks] Discuss the results for the Yukawa matrices from part (d) (for both cases (i) and (ii)) in view of matching the physical masses of the leptons. (The electron mass equals $\sim 0.511\,\mathrm{MeV}$, the muon mass $\sim 105.7\,\mathrm{Mev}$ and the tau mass $\sim 1777\,\mathrm{MeV}$.)

2. (a) [4 marks] Write down the Dynkin diagram, the Cartan matrix and the metric tensor for $A_2 = \mathrm{su}(3)_{\mathbb{C}}$.

(b) [6 marks] Irreducible representations (irreps) of A_2 are labelled in terms of their highest weight with Dynkin label (a_1, a_2). Derive formulae for the dimension, the quadratic Casimir and the index of A_2 representations in terms of (a_1, a_2), starting with the Weyl formula and the general formula for the quadratic Casimir.

(c) [4 marks] Find all the A_2 irreps with dimension less than 10 in terms of their highest weight Dynkin label and compute their dimensions, quadratic Casimirs and indices.

(d) [6 marks] The (one-loop) β-function for a gauge theory with scalar matter is given by

$$\beta(g) = -\frac{1}{32\pi^2}\left[\frac{11}{3}c(\mathrm{ad}) - \frac{1}{6}c(r_S)\right] g^3 \;,$$

where g is the gauge coupling, $c(\cdot)$ denotes the index of a representation, ad the adjoint representation and r_S the representation of the scalar matter. Assume a gauge group $SU(3)$ and a scalar matter representation r_S which is given by (i) n copies of the fundamental representation and (ii) n copies of the second rank symmetric tensor representation. In either case, what is the maximal number n for which the theory is still asymptotically free, as judged by the above one-loop beta function (that is, the value of g decreases with increasing energy)?

(e) [5 marks] Consider an $SU(3)$ gauge theory with (Weyl) fermions in irreps r_i, where $i = 1, \ldots, \nu$, with an additional $\mathbb{Z}_n$ symmetry under which the fermions in r_i carry charge q_i. The discrete $\mathbb{Z}_n$ gauge anomaly is defined by $\mathcal{A} = \sum_{i=1}^{\nu} q_i c(r_i)$ and the theory is said to be anomaly-free if $\mathcal{A} = 0 \bmod n$. (i) For the case of two irreps, $\nu = 2$, and $r_1 = r_2 = r_F$, where F is the fundamental representation, find the charge choices q_1, q_2 for which the theory is anomaly-free. (ii) Do the same for two irreps $r_1 = r_F$ and $r_2 = r_{S^2F}$ (the second rank symmetric tensor of the fundamental), focusing on the symmetry $\mathbb{Z}_4$.

3. (a) [4 marks] Let G be a group and $H \subset G$ a subgroup. The commutant of H in G is defined as $C_G(H) = \{g \in G \,|\, gh = hg \; \forall h \in H\}$. Show that $C_G(H)$ is a subgroup of G.

   (b) [6 marks] Consider the group emdedding

$$H := \mathrm{SU}(n_1) \times \mathrm{SU}(n_2) \ni (U_1, U_2) \mapsto \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \in \mathrm{SU}(n_1 + n_2) =: G \; ,$$

   find the commutant $C_G(H)$ and show it is isomorphic to a $U(1)$ symmetry. (Hint: Write elements of G as block-matrices and use Schur's Lemma.) Specialise the result to $n_1 = 2$ and $n_2 = 3$ and denote the $U(1)$ symmetry isomorphic to $C_G(H)$ by $U_c(1)$.

   (c) [7 marks] For $H = \mathrm{SU}(2) \times \mathrm{SU}(3)$ embedded into $G = \mathrm{SU}(5)$ as in (b) find the branching of the fundamental $\mathbf{5}$, the complex conjugate fundamental $\bar{\mathbf{5}}$ and the second rank anti-symmetric tensor $\mathbf{10}$ of G under H, using tensor methods. Also, determine how each of the resulting H representations transforms under $U_c(1)$.

   (d) [4 marks] Find how the adjoint of $G = \mathrm{SU}(5)$ branches under $H = \mathrm{SU}(2) \times \mathrm{SU}(3)$. Compare two gauge theories with groups G and H. What are the H representations of the gauge fields of G which are not gauge fields of $H \times U_c(1)$?

   (e) [4 marks] The decomposition of the SU(5) representation $\bar{\mathbf{5}} \oplus \mathbf{10}$ under $H \times U_c(1)$ leads to one family in the standard model of particles. Show that this cannot be accomplished by starting with the $\mathbf{15}$ representation of SU(5) (the rank two symmetric tensor).

4. (a) [4 marks] Write down the Dynkin diagram and the Cartan matrix for $A_5 = \mathrm{su}(6)_{\mathbb{C}}$. Why is $A_4 = \mathrm{su}(5)_{\mathbb{C}}$ a sub-algebra of A_5?

   (b) [5 marks] For both A_5 and A_4 consider the fundamental, the complex conjugate of the fundamental, the second rank anti-symmetric tensor of the fundamental and the second rank anti-symmetric tensor of the complex conjugate fundamental and write down the highest weight Dynkin label, the Young tableau, the dimension of the representation and the tensor in each case.

   (c) [6 marks] Work out the weight systems of all the representations from part (b).

   (d) [6 marks] For all A_5 representations in part (c), use the weight systems to determine their branching into A_4 representations.

   (e) [4 marks] Which A_5 representations or direct sum of representations from the ones mentioned in (b) can be used to accommodate one family of quarks and leptons in the context of an SU(6) grand unified theory? Which additional $SU(5)$ multiplets arise?