Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

GROUPS AND REPRESENTATION

Hilary Term 2021

FRIDAY, 15TH JANUARY 2020, Opening Time: 09:30 a.m GMT

You should submit answers to ***three*** *out of the four questions.*

You have **3 hours** *writing time to complete the paper and up to* **1 hour** *technical time for uploading your file. The allotted technical time must not be used to finish writing the paper. The use of a calculator and/or computer algebra packages is* **not** *allowed.*

1. (a) [5 marks] Consider a representation $R : G \to \mathrm{GL}(V)$ of a finite group G on a complex vector space V with character χ. Explain how R induces representations, denoted S^2R and $\wedge^2 R$, on the symmetric and anti-symmetric tensor products S^2V and $\wedge^2 V$, respectively. Show that their characters χ_{S^2R} and $\chi_{\wedge^2 R}$ satisfy

$$\chi_{S^2R}(g) = \frac{1}{2}(\chi(g)^2 + \chi(g^2)) , \qquad \chi_{\wedge^2 R} = \frac{1}{2}(\chi(g)^2 - \chi(g^2)) ,$$

for all $g \in G$. (Hint: Think about the eigenvalues of a representation matrix $R(g)$.)

(b) [5 marks] Compute the elements of the finite group G generated by the Pauli matrices σ_1 and σ_3. What is the order of this group and what are its conjugacy classes? How many complex irreducible representations does G have and what are their dimensions?

(c) [5 marks] Denote by $\rho : G \to \mathrm{GL}(\mathbb{C}^2)$ the two-dimensional representation given by the matrices which define the group G from part (b). Compute the character χ_ρ of this representation and show that ρ is irreducible. Write down the character table of G.

(d) [5 marks] Work out the characters of the representations $S^2\rho$ and $\wedge^2\rho$ and determine their Clebsch-Gordan decompositions.

(e) [5 marks] The group G from (b) acts on vectors $v = (x, y)^T \in \mathbb{C}^2$ as $v \mapsto gv$ for $g \in G$. Are there G-invariant quadratic polynomials in x and y? If so, write them down explicitly.

2. Consider the permutation group S_3 which consists of bijective maps $\{1, 2, 3\} \to \{1, 2, 3\}$.

(a) [6 marks] How many complex irreducible representations does S_3 have and what are their dimensions? Find the character table of S_3. In each case provide reasoning for your answer based on general properties of finite groups and their representations. Work out the Clebsch-Gordan decompositions of all possible tensor products $R_i \otimes R_j$, where R_i are the irreducible representations.

(b) [4 marks] Let e_k, for $k = 1, 2, 3$, be the three-dimensional standard unit vectors. Show that the map $R : S_3 \to \mathrm{GL}(\mathbb{C}^3)$ defined by $R(\sigma)(e_k) = e_{\sigma(k)}$ is a representation. Determine the irreducible representations contained in R.

(c) [6 marks] For the representation R in (b) find the R-invariant vector subspaces $U \subset \mathbb{C}^3$ on which the irreducible representations within R act. For each U, choose a basis and find the explicit representation matrices $R_U(\tau)$ relative to this basis for all transpositions $\tau \in S_3$.

(d) [6 marks] Consider Yukawa terms of the form $\sum_{i,j=1}^{3} \lambda_{ij} H \bar{\psi}_L^i \psi_R^j$, where H is a scalar, ψ_L^i and ψ_R^i are left- and right-handed fermions and $i, j = 1, 2, 3$ are family indices. Assign to H the trivial S_3 representation, to each of ψ_L^3 and ψ_R^3 a one-dimensional but non-trivial S_3 representation, and to each of (ψ_L^1, ψ_L^2) and (ψ_R^1, ψ_R^2) a two-dimensional irreducible representation of S_3. Given this assignment of representations, determine the most general S_3-invariant Yukawa couplings λ_{ij}.

(e) [3 marks] Assume that the fermions ψ^i are, in fact, the electron, the muon and the tau. Discuss to what extent the Yukawa couplings λ_{ij} obtained in part (d) can accommodate their physical masses. (The electron mass equals $\sim 0.511\,\mathrm{MeV}$, the muon mass $\sim 105.7\,\mathrm{Mev}$ and the tau mass $\sim 1777\,\mathrm{MeV}$.)

3. (a) [4 marks] The Dynkin diagram of A_2 is $\circ - \circ$. Write down the Cartan matrix of A_2 and work out the weight systems of the representations with highest weights $(1, 0)$, $(0, 1)$ and $(1, 1)$.

   (b) [3 marks] In the context of the $SU(3)$ quark model, the embedding of electrical charge into $SU(3)$ is described by the vector $Q = \frac{1}{3}(2, 1)$ (in the dual basis), while hypercharge is embedded via $Y = \frac{1}{3}(1, 2)$. Find the charge/hypercharge pairs for the mesons.

   (c) [6 marks] The Dynkin diagram of G_2 is $\circ \equiv \bullet$. Write down the Cartan matrix and find the weight systems of the representations with highest weights $(0, 1)$ and $(1, 0)$.

   (d) [6 marks] Argue from (extended) Dynkin diagrams that A_2 is a sub-algebra of G_2. The associated projection matrix which maps G_2 weights to A_2 weights is given by

$$P = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} .$$

   Determine the branching of the G_2 representations from part (c) into A_2 representations.

   (e) [6 marks] Discuss a possible extension of the $SU(3)$ quark model to G_2, focusing on the mesons. Which G_2 representation would the known mesons have to reside in? Determine the vectors (in the dual basis) which describe the embedding of charge and hypercharge into G_2 such that all known mesons retain their physical charges. How many new mesons does the G_2 model predict and what are their charges and hypercharges?

4. (a) [8 marks] For the group $SU(4)$, write down the Young tableaux, dimensions and tensors for the representations with highest-weight Dynkin labels $(1, 0, 0)$ and $(0, 1, 0)$, $(0, 0, 1)$ and $(1, 0, 1)$. Write down the $SU(4)$ transformations for these tensors. Also, work out the tensor products $(1, 0, 0) \otimes (1, 0, 0)$ and $(1, 0, 0) \otimes (0, 0, 1)$ using Young tableaux.

   (b) [6 marks] Consider the $SU(4)$ subgroups $H_2 \cong SU(2) \times SU(2)$ and $H_3 \cong SU(3)$, embedded as

$$H_2 \ni U_2 = \begin{pmatrix} V & 0 \\ 0 & \tilde{V} \end{pmatrix} \in SU(4) , \quad \text{where} \quad V, \tilde{V} \in SU(2) ,$$

$$H_3 \ni U_3 = \begin{pmatrix} W & 0 \\ 0 & 1 \end{pmatrix} \in SU(4) , \quad \text{where} \quad W \in SU(3) .$$

   Find the branching of the $SU(4)$ representations with highest-weight Dynkin labels $(1, 0, 0)$, $(0, 1, 0)$ and $(1, 0, 1)$ under each of these subgroups.

   (c) [4 marks] Consider a scalar field ϕ which transforms under an $SU(4)$ representation R. The scalar ϕ has an $SU(4)$-invariant scalar potential which leads to a symmetry-breaking vacuum expectation value $\langle \phi \rangle$. Choose for R each of the representations from part (b) in turn and use the results for their branching under H_2, H_3 to determine which of the symmetry breaking patterns $SU(4) \to H_2$, $SU(4) \to H_3$ are allowed and which are forbidden.

   (d) [7 marks] For the cases not excluded in part (c), find an appropriate form of $\langle \phi \rangle$ which leaves the subgroup H_2 or H_3 unbroken. In each case, discuss whether or not H_2 or H_3 are unbroken for generic choices of $\langle \phi \rangle$.