A15090W1

PUBLIC EXAMINATION

Honour School of Mathematical and Theoretical Physics (MMathPhys)

Master of Science in Mathematical and Theoretical Physics (MScMTP)

Groups and Representations

FRIDAY, 17TH JANUARY 2020, 09:30 am to 12:30 pm

Answer **three** *out of four questions.*

Start the answer to each questions on a new page.
Calculators are not allowed.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do NOT turn over until told that you may do so.

1.)

(a) Define the terms "group representation", "equivalent representations", "reducible representation" and "irreducible representation". What does "Schur's Lemma" state? [7 marks]

(b) Let $R_i : G \to \mathrm{Gl}(\mathbb{C}^{m_i})$, where $i = 1, 2, 3$, be three irreducible representations of a group G. The representation $R : G \to \mathrm{Gl}(\mathbb{C}^m)$, where $m = m_1 + m_2 + m_3$, is defined by the block matrix

$$R(g) = \begin{pmatrix} R_1(g) & 0 & 0 \\ 0 & R_2(g) & 0 \\ 0 & 0 & R_3(g) \end{pmatrix} .$$

Provided that the representations R_1, R_2, R_3 are pairwise inequivalent, what is the form of the most general $m \times m$ matrix P (with complex entries) that commutes with $R(g)$ for all $g \in G$? [7 marks]

(c) Now assume that $R_1 = R_2$ while R_1 and R_3 are inequivalent. Otherwise the set-up is the same as in part (b). What is the most general form of the matrix P in this case? [6 marks]

(d) Assuming that $R_1 = R_2 = R_3$, but otherwise keeping the set-up as in part (b), what is the most general form of P now? [5 marks]

2.) A group G is generated by the matrices

$$g_1 = \begin{pmatrix} 0 & \alpha^* \\ \alpha & 0 \end{pmatrix} , \qquad g_2 = g_1^* , \qquad g_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} ,$$

where $\alpha = \exp(2\pi i/3)$.

(a) Show that G is a group of order 6 and that is it isomorphic to the permutation group S_3, the group of bijective maps $\{1, 2, 3\} \to \{1, 2, 3\}$. Write down the conjugacy classes of G.

[6 marks]

(b) Find all irreducible representations and the character table of G. [6 marks]

(c) Let e_i, where $i = 1, 2, 3$, be the standard unit basis vectors in $\mathbb{C}^3$. A representation $R : S_3 \to \mathrm{Gl}(\mathbb{C}^3)$ is defined by $R(\sigma)(e_i) = e_{\sigma(i)}$. Find the irreducible representation content of R.

[6 marks]

(d) Consider the vector space

$$V = \left\{ \sum_{i,j=1}^{3} c_{ij} x_i x_j \,|\, c_{ij} \in \mathbb{C} \right\}$$

of quadratic polynomials in x_1, x_2, x_3. A representation $\tilde{R} : S_3 \to \mathrm{Gl}(V)$ is induced by permuting the coordinates x_i, that is $\tilde{R}(\sigma)$ acts on a quadratic polynomial simply by permuting the coordinates x_i on which it depends as $x_i \mapsto x_{\sigma(i)}$. What is the irreducible representation content of $\tilde{R}$? Find the sub-space of singlets V^{S_3} in V. [7 marks]

3.)

(a) For groups G_1 and G_2 show that $G := G_1 \times G_2 = \{(g_1, g_2) \,|\, g_1 \in G_1,\ g_2 \in G_2\}$ with multiplication $(g_1, g_2)(\tilde{g}_1, \tilde{g}_2) := (g_1\tilde{g}_1, g_2\tilde{g}_2)$ forms a group. For representations $R_1 : G_1 \to \mathrm{Gl}(\mathbb{C}^{n_1})$ and $R_2 : G_2 \to \mathrm{Gl}(\mathbb{C}^{n_2})$ show that $R : G \to \mathrm{Gl}(\mathbb{C}^{n_1 n_2})$ defined by $R(g_1, g_2) = R_1(g_1) \times R_2(g_2)$ is a representation. (You can use properties of the Kronecker product $\times$ without proof.) [6 marks]

(b) For the group $SU(3)$, write down the Young tableaux, the associated tensors, the highest weight Dynkin labels and the dimensions for the six irreducible representations with the smallest dimensions. [6 marks]

(c) Consider the group $G = SU(3) \times SU(3)$ and its sub-group $H = SU(3) \times SU(2)$, where the first $SU(3)$ factors in G and H are identical and the $SU(2)$ factor in H is embedded into the second $SU(3)$ factor of G as

$$SU(2) \ni U \hookrightarrow \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix} \in SU(3) \; .$$

Denote representations of G by a pair (R_1, R_2), where R_1 and R_2 are representations of the first and second $SU(3)$ factor, respectively. Find the branching of the G representations $(\mathbf{3}, \mathbf{3})$, $(\bar{\mathbf{3}}, \mathbf{1})$, $(\mathbf{1}, \mathbf{3})$ and $(\mathbf{1}, \mathbf{8})$ under the sub-group H. [5 marks]

(d) Find the $U(1)$ within G which commutes with H. For this $U(1)$, determine the charges of the H representations found in part (c). [5 marks]

(e) Is G a suitable unifying group for the Standard Model of particle physics? Discuss. [3 marks]

4.)

(a) Find the Lie algebra $su(4)$ of the group $SU(4)$ and write down a simple basis for this algebra. What is the dimension and the rank of $su(4)$? [4 marks]

(b) Write down the Cartan matrix and the Dynkin diagram for $su(4)_{\mathbb{C}}$. Find the weights (in the Dynkin basis) for the representations with highest weights $(1, 0, 0)$, $(0, 0, 1)$ and $(1, 0, 1)$. What are the dimensions of these representations? (You can assume that the weights of the representations with highest weights $(1, 0, 0)$ and $(0, 0, 1)$ are all non-degenerate.) [7 marks]

(c) An embedding of $su(3)_{\mathbb{C}}$ into $su(4)_{\mathbb{C}}$ is defined by the projection matrix

$$P = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} .$$

Find the branching of the representations from part (b) for this $su(3)_{\mathbb{C}}$ sub-algebra. [5 marks]

(d) Suppose you want to construct a quark model for four quarks (u, d, s, c) based on $SU(4)$, with the $SU(3)$ sub-group identified in part (c) the usual symmetry group for (u, d, s). Which $SU(4)$ representation describes the mesons in this quark model and what is its $SU(3)$ representation content? [5 marks]

(e) Write the additional mesons in the $SU(4)$ model which are not contained in the $SU(3)$ quark model in terms of (u, d, s, c). [4 marks]