A15090W1

PUBLIC EXAMINATION

Honour School of Mathematical and Theoretical Physics (MMathPhys)

Master of Science in Mathematical and Theoretical Physics (MScMTP)

Groups and Representations

FRIDAY, 12TH JANUARY 2018, from 09:30am to 12:30pm

Answer **three** *out of four questions.*

Start the answer to each questions on a new page.
Calculators are not allowed.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do NOT turn over until told that you may do so.

1.)

(a) Write down the definitions of a group, an Abelian group and a sub-group. Using Schur's Lemma, prove that the complex, irreducible representations of Abelian groups are one-dimensional. [6]

(b) Write down the complex, irreducible representations of $\mathbb{Z}_n$. [3]

(c) Write down the character table of $\mathbb{Z}_3 = \{0, 1, 2\}$ and explicitly check that the characters of the complex, irreducible representations of $\mathbb{Z}_3$ are ortho-normal. [4]

(d) Let $\alpha = \exp(2\pi i/3)$ and

$$A = \frac{1}{2}\begin{pmatrix} 1+\alpha & 1-\alpha \\ 1-\alpha & 1+\alpha \end{pmatrix} .$$

Show that $R(g) := A^g$, where $g \in \mathbb{Z}_3 = \{0, 1, 2\}$, defines a representation of $\mathbb{Z}_3$. Which irreducible $\mathbb{Z}_3$ representations does R contain? [5]

(e) Let $V = \text{Span}(x^2, y^2, z^2, xy, xz, yz)$ be the vector space of quadratic polynomials in three variables x, y, z. On this vector space, define a $\mathbb{Z}_3$ representation $\tilde{R}$ by demanding that $\tilde{R}(1)$ cyclically permutes the coordinates x, y, z, so $x \to y$, $y \to z$ and $z \to x$ (so that, for example, $\tilde{R}(1)(xy) = yz$). Work out the character of this representation and its irreducible representation content. [7]

2.) Define the two matrices

$$g_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} , \qquad g_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} .$$

A finite group G of order 8 is given by the matrices

$$G = \{g_0 = e = \mathbb{1}_3,\ g_1,\ g_2,\ g_3 = g_2^2,\ g_4 = g_1g_2,\ g_5 = g_2g_1,\ g_6 = g_1g_2g_1,\ g_7 = g_2^2g_1\} .$$

(a) Show that G has the five conjugacy classes $C_0 = \{e\}$, $C_1 = \{g_3\}$, $C_2 = \{g_1, g_7\}$, $C_3 = \{g_2, g_6\}$ and $C_4 = \{g_4, g_5\}$. Show that G is a sub-group of $SU(3)$. [6]

(b) Find all the complex, irreducible representations of G and write down the character table. [10]

(c) Since G is a matrix group it is represented by itself. Call this representation R. Which irreducible representations do R and $R \otimes R$ contain? [4]

(d) Work out the branching of the $SU(3)$ representations $\mathbf{3}$, $\bar{\mathbf{3}}$ and $\mathbf{8}$ under the group G. [5]

3.) Consider the three-dimensional Lorentz group G, that is, the group of real 3×3 matrices Λ satisfying $\Lambda^T \eta \Lambda = \eta$, where $\eta = \text{diag}(-1, 1, 1)$.

(a) Find the Lie algebra $\mathcal{L}(G)$ of G and determine its dimension and rank. Write down a simple basis of matrices T_i for $\mathcal{L}(G)$. [5]

(b) Show that (the complexification of) $\mathcal{L}(G)$ is isomorphic to (the complexification of) $su(2) = \mathcal{L}(SU(2))$. What does this imply for the representations of $\mathcal{L}(G)$ and their dimensions? [7]

(c) Explicitly construct the two-dimensional irreducible representation r of $\mathcal{L}(G)$. [4]

(d) Define the matrices γ_μ, where $\mu = 0, 1, 2$, by $\gamma_0 := -i\sigma$, $\gamma_1 := \sigma_2$ and $\gamma_2 := \sigma_3$, where σ_i are the Pauli matrices. Show that these matrices γ_μ form a viable set of gamma matrices in three dimensions. [3]

(e) Consider spinors $\psi \in \mathbb{C}^2$ which transform under the two-dimensional representation r from part (c), that is, $\delta\psi = r(T)\psi$ for $T \in \mathcal{L}(G)$, and define $\bar{\psi} := \psi^\dagger \gamma_0$. Show that the expression $\bar{\psi}\psi$ is invariant under infinitesimal Lorentz transformations. [6]

4.) Consider the Lie algebras $B_3 \sim so(7)$ and $A_3 \sim su(4)$.

(a) Draw the Dynkin diagrams for B_3 and A_3 and write down their Cartan matrices. Use the extended Dynkin diagram of B_3 to argue that A_3 a sub-algebra of B_3. [5]

(b) Construct the weight systems of the B_3 representations with highest weights $(1, 0, 0)$ and $(0, 0, 1)$. (You can assume that all weights are non-degenerate.) [6]

(c) The projection matrix from B_3 to A_3 is given by

$$P(A_3 \subset B_3) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} .$$

Use this matrix to determine the branching under A_3 of the B_3 representations constructed in part (b). Write down the Young tableaux and tensors for the A_3 representations which arise in these branchings. [8]

(d) Suppose that $SO(7)$ is part of unified group such that the colour $SU(3)$ group is embedded as $SU(3) \subset SU(4) \subset SO(7)$. Further assume this $SO(7)$ theory has particles in the two representations in part (b). Determine the $SU(3)$ multiplets this leads to and discuss their physical interpretation. [6]