A15090W1

PUBLIC EXAMINATION

Honour School of Mathematical and Theoretical Physics Part C

Master of Science in Mathematical and Theoretical Physics

GROUPS AND REPRESENTATIONS

Hilary Term 2016

FRIDAY, 15th JANUARY 2016, 9:30am to 12:30pm

You should submit answers to three out of four questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so.

1.) Let G be a group.

a) Define the terms "representation of a group", "irreducible representation", "faithful representation" and "unitary representation". [5]

b) State Schur's Lemma for an irreducible representation $R : G \to \mathrm{Gl}(V)$ on a complex vector space V and a linear map $P : V \to V$. Use Schur's Lemma to show that a complex irreducible representation of an Abelian group must be one-dimensional. [6]

c) Consider the Abelian group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ which consists of $\{(0,0),(1,0),(0,1),(1,1)\}$, with the group operation being addition modulo 2. Write down the irreducible complex representations and the character table for this group. [6]

d) Let $V = \{ax^2 + bxy + cy^2 \,|\, a,b,c \in \mathbb{C}\}$ be the vector space of quadratic polynomials p in two variables x, y with complex coefficients. A map $R : \mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathrm{Gl}(V)$ is defined by $R((1,0))(p)(x,y) := p(x,-y)$, $R((0,1))(p)(x,y) := p(y,x)$ and $R((1,1)) := R((1,0))R((0,1))$. Why is R a representation? Work out the representation matrices for $R((1,0))$ and $R((0,1))$ relative to the standard monomial basis $\{x^2, xy, y^2\}$ of V. Find the character and the irreducible representation content of R. [8]

2.) The quaternion group Q can be defined as a matrix group with the eight elements

$$Q = \{\pm\mathbb{1}_2, \pm i\sigma_1, \pm i\sigma_2, \pm i\sigma_3\} \ ,$$

where σ_i are the Pauli matrices, explicitly given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \ .$$

a) Find the conjugacy classes of Q. How many irreducible, complex representations does Q have and what are the dimensions of these representations? [5]

b) Given the two one-dimensional representations

$$\begin{array}{llll} R_1(\pm\mathbb{1}_2) = 1 & R_1(\pm i\sigma_1) = 1 & R_1(\pm i\sigma_2) = -1 & R_1(\pm i\sigma_3) = -1 \\ R_2(\pm\mathbb{1}_2) = 1 & R_2(\pm i\sigma_1) = -1 & R_2(\pm i\sigma_2) = 1 & R_2(\pm i\sigma_3) = -1 \end{array}$$

of Q, write down the character table of Q and the remaining irreducible representation or representations. [8]

c) A four-dimensional representation R_4 of Q is given by $R_4(\pm\mathbb{1}_2) = \pm\mathbb{1}_4$ and

$$R_4(\pm i\sigma_1) = \pm\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \ , \quad R_4(\pm i\sigma_2) = \pm\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$R_4(\pm i\sigma_3) = \pm\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

Verify that R_4 is indeed a representation of Q and determine its irreducible representation content. [6]

d) What is the irreducible representation content of $R_4 \oplus R_4$ and $R_4 \otimes R_4$? [6]

3.) Consider the group $SU(4)$ of 4×4 special unitary matrices.

a) Determine the Lie algebra and the Cartan sub-algebra of $SU(4)$. What are the dimension and the rank of this Lie algebra? Write down a simple basis for the Cartan sub-algebra. [6]

b) For the fundamental representation, $\mathbf{4}$, of $SU(4)$, find the weights of the standard unit vectors $\mathbf{e}_i$, where $i = 1, \ldots, 4$, in $\mathbb{C}^4$. What are the weights of the complex conjugate of the fundamental representation, $\bar{\mathbf{4}}$? [4]

c) Using Young tableaux, find the irreducible representations in $\mathbf{4} \otimes \mathbf{4}$ and $\mathbf{4} \otimes \bar{\mathbf{4}}$. For $\mathbf{4} \otimes \mathbf{4}$, identify the tensors associated to these irreducible representations. [8]

d) Consider the $SU(3)$ sub-group of $SU(4)$ defined by the embedding

$$U = \begin{pmatrix} U_3 & 0 \\ 0 & 1 \end{pmatrix} ,$$

where $U_3 \in SU(3)$. How do the representations $\mathbf{4}$, $\bar{\mathbf{4}}$ and $\mathbf{4} \otimes \mathbf{4}$ branch under this $SU(3)$ sub-group? [7]

4.) Consider the group $SO(7)$ of 7×7 special orthogonal matrices.

a) Determine the Lie algebra, the Cartan sub-algebra, the dimension and the rank for this group. [5]

b) The Cartan matrix for the associated algebra B_3 is given by

$$A(B_3) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -1 & 2 \end{pmatrix} .$$

Find the weight system for the representation with highest Dynkin weight $(1, 0, 0)$. [7]

c) Find the weight system for the representation with highest Dynkin weight $(0, 0, 1)$. [7]

d) The algebra A_3, associated to $SU(4)$, can be embedded into B_3 via the projection matrix

$$P(A_3 \subset B_3) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} .$$

Using this matrix, determine the branching under $SU(4)$ of the $SO(7)$ representation with highest weight $(0, 0, 1)$ from part **4.**) c). You can use the Cartan matrix of A_3 given by

$$A(A_3) = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix} .$$

[6]