Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

GEOPHYSICAL FlUID DYNAMICS

Trinity Term 2022

Wednesday 8th June 2022, 9:30am to 11:30am

You should submit answers to both questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

You are permitted to use the following material(s):
A4 summary sheet Calculator

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. (a) [5 marks] The flow in certain regions of the upper ocean may be modeled by the following approximation to the reduced-gravity equations on an f-plane:

$$\begin{aligned}
\frac{\partial u}{\partial t} - f_0 v + g' \frac{\partial h}{\partial x} &= 0, \\
\frac{\partial v}{\partial t} + f_0 u + g' \frac{\partial h}{\partial y} &= 0, \\
\frac{\partial h}{\partial t} + h_0 \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} \right) &= 0,
\end{aligned}$$

where the symbols have their usual meanings. Briefly outline the assumptions underlying these equations and give a physical interpretation of each one.

(b) [6 marks] Show that a wave-like solution exists adjacent to a north-south boundary (oriented parallel to longitude circles) of the form

$$h = h_0 + A(x) \exp[i(\omega_K t - \ell_K y)]; \quad u = 0,$$

where h_0 is a constant, ω_K is the frequency and ℓ_K the meridional wavenumber. Obtain expressions for $A(x)$ and the phase speed of waves of meridional wavenumber ℓ_K in terms of g', h_0 and f_0, showing that they remain trapped close to the boundary. Deduce their direction of propagation in the vicinity of a steep, eastern boundary to an ocean basin in the northern hemisphere. What, if anything, would change in the southern hemisphere?

(c) [5 marks] A disturbance in the thermocline is incident from the west onto an eastern boundary, aligned due north-south close to the equator, which launches waves in h and v moving polewards. Given the Rossby radius of deformation $L_d = \sqrt{g'h_0}/f_0 \simeq 100$ km at latitude $\phi = 10°$ N or S, estimate the speed and direction of propagation of such waves and their characteristic length scale in longitude, assuming they are dynamically similar to the coastal waves mentioned above. Sketch the structure of such waves in h and horizontal velocity.

You are given $\sin 10° = 0.17$ *and* $\cos 10° = 0.98$.

(d) [9 marks] Show that the meridional propagation of long wavelength ($\ell_K \ll 1/L_d$) coastal waves could excite a train of long zonal wavelength, midlatitude Rossby waves, propagating westwards into the open ocean from the coastal boundary, provided $|\omega_K| \simeq \beta k_R g' h_0 / f_0^2$, given their dispersion relation is of the form

$$\omega_R = \frac{-\beta k_R}{k_R^2 + \ell_R^2 + 1/L_d^2},$$

where β is the northward gradient of Coriolis parameter f. Estimate how long it would take for such waves to cross the equatorial Pacific Ocean around a latitude of $\phi = 10°$ N or S, and comment on the possible significance of this result for interannual climate variations.

2. (a) [7 marks] The equations of motion for quasigeostrophic dynamics are

$$\left(\frac{\partial}{\partial t} + \mathbf{u}_g \cdot \nabla\right) Q = 0, \qquad Q = \frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial \psi}{\partial z}\right) + f_0 + \beta y.$$

Give a physical interpretation of each term in these equations, and briefly outline the assumptions required to derive them. How is $\mathbf{u}_g = (u, v, 0)$ related to ψ?

If gradients of density are negligible in a fluid layer of constant depth, justify why there are solutions with $\psi = \psi(x, y)$ depending only on horizontal position.

(b) [3 marks] Show that

$$\int_{\mathbf{x}_0}^{\mathbf{x}_1} \mathbf{u}_g \cdot \mathbf{n} \, \mathrm{ds} = \psi(\mathbf{x}_0) - \psi(\mathbf{x}_1),$$

for any contour joining $\mathbf{x}_0 = (x_0, y_0)$ to $\mathbf{x}_1 = (x_1, y_1)$, where $\mathbf{n}$ is the normal to the contour in the (x, y) plane.

(c) [9 marks] Consider an idealised model of river discharge into an estuary, which takes the form of a long channel of constant depth H and constant width L for $x > 0$ and $0 < y < L$. Density gradients can be neglected. The river discharges fluid into the channel with zero relative vorticity, with a volume input rate per unit depth F at the corner $(0, L)$. The boundaries $x = 0$, $y = 0$ and $y = L$ are otherwise impermeable. There is negligible relative vorticity far downstream as $x \to \infty$. The gravitational restoring of the upper free surface is sufficiently strong that it can be approximated as a planar rigid lid at $z = 0$. The flow can be treated as on an f-plane in the Northern hemisphere.

Justify that we can use boundary conditions

$$\psi(0, y) = 0, \qquad \psi(x, 0) = 0, \qquad \psi(x, L) = -F,$$

to describe this flow. Assuming that the flow dynamics are quasigeostrophic, show that this problem has a solution

$$\psi = -F\frac{y}{L} + \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi y}{L}\right) \mathrm{e}^{\gamma_n x},$$

for some B_n and γ_n to be determined.

You may quote without proof the result

$$\int_0^L y \sin\left(\frac{n\pi y}{L}\right) dy = (-1)^{n+1}\frac{L^2}{n\pi}.$$

(d) [6 marks] Sketch the approximate leading order variation of the zonal velocity $u(L, y)$ as a function of distance y across the width of the channel at $x = L$. Hence sketch the approximate pattern of streamlines as a function of x and y. Discuss the force balances that maintain this pattern of flow.