A15274W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

GEOPHYSICAL FlUID DYNAMICS

Trinity Term 2021

TUESDAY, 8TH JUNE 2021, Opening Time 09:30 am UK Time

You should submit answers to both questions.

You have **2 hours** *writing time to complete the paper and up to* **30 minutes** *technical time for uploading your file. The allotted technical time must not be used to finish writing the paper.*
Mode of completion: **handwritten**
You are permitted to use the following material(s):
Calculator (candidate to provide)
Formula Sheet (provided by course administrator prior to the exam)
The use of computer algebra packages is **not** *allowed.*

1. Steady, horizontal flow in a barotropic ocean or atmosphere near the equator (where $y = 0$), subject to zonally symmetric forcing of the zonal flow (e.g., due to wind-induced surface stresses and friction), can be modelled by the equations

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\ -\beta y v &= -\frac{\partial \Phi}{\partial x} - ru + F_u(y), \\ \beta y u &= -\frac{\partial \Phi}{\partial y} - rv, \end{aligned}$$

where u, v are the horizontal velocity components and β is the northward planetary vorticity gradient.

(a) [5 marks] Give a brief physical interpretation of these equations and show that they lead to a vorticity equation of the form

$$\beta \frac{\partial \psi}{\partial x} + r\nabla_{\mathrm{h}}^2 \psi = F_\zeta(y),$$

where ∇_{h}^2 is the horizontal Laplacian. Define r, and define how ψ and F_ζ are determined in terms of u, v and F_u.

(b) [5 marks] An ocean is subject to a purely zonal vorticity forcing of the form

$$F_\zeta = F_0 \cos(\pi y/b),$$

where F_0 and b are positive constants. Obtain an expression for the steady northward flow v that results from this forcing in the absence of friction. Hence derive an expression for the eastward flow u, assuming that the ocean has an eastern boundary at $x = L$. Why does such a flow not occur if there is also a western boundary at $x = 0$?

(c) [10 marks] Sputnik Planitia is a large, roughly rectangular basin, centred on the equator at the surface of the dwarf planet Pluto and bounded by high mountain ridges at both its eastern and western boundaries at $x = 0$ and $x = L$. Assume that we can treat the flow inside the basin as barotropic and subject to a zonally symmetric forcing given by F_ζ defined above. For sufficiently small friction r, show that the streamfunction for the horizontal flow in this case is approximately given by

$$\psi \simeq A\left(1 - \gamma L\,\mathrm{e}^{-\lambda x} - \mathrm{e}^{\gamma(x-L)}\right)\cos(\pi y/b),$$

where λ, γ are constants with $\lambda L \gg 1$ and $\gamma L \ll 1$. Obtain approximate expressions for λ and γ accurate to leading order in r, and express A in terms of F_0, r and b. Give rough sketches (i) of the streamlines within the domain $-b/2 \leqslant y \leqslant b/2$ and $0 \leqslant x \leqslant L$, and (ii) the profile of the northward velocity $v(x)$ along $y = 0$.

(d) [5 marks] Estimate the width of the western boundary current, given that $r = 1$ (Pluto day)$^{-1}$ and b corresponds to 60° latitude on Pluto. Also estimate the peak magnitude and direction of the north-south velocity component v (a) in the open basin and (b) in the western boundary current, given that $F_0 = 10^{-10}\,\mathrm{s}^{-2}$ and L corresponds to 60° in longitude. (Pluto's radius $a = 1188\,\mathrm{km}$ and its rotation period is 6.4 Earth days).

2. (a) [4 marks] Consider an initial state consisting of an isolated cyclonic vortex in a fluid that is otherwise at rest but subject to a meridional gradient in the Coriolis parameter $\beta = df/dy > 0$. Explain, using sketches, how the flow develops according to the Rossby-wave propagation mechanism.

(b) [5 marks] The simplest system capable of supporting Rossby waves is incompressible flow in a single layer of fixed depth on a (midlatitude) β-plane, represented by

$$\left(\frac{\partial}{\partial t} + u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\xi + \beta v = 0,$$

where ξ is the relative vorticity. Linearise this equation about a background constant westerly wind $\bar{u}$ and show that it supports wave solutions with dispersion relation

$$\omega = \bar{u}k - \frac{\beta k}{k^2 + l^2},$$

where ω is the frequency and k and l are the zonal and meridional wavenumbers, respectively.

(c) [6 marks] For the situation where $\bar{u} = 0$ and $\omega \geqslant 0$, sketch a dispersion diagram in the (k, l) plane, showing contours of constant ω and also arrows indicating the group velocity. Determine the regions of this plane where the zonal phase and group velocities are directed eastward or westward.

(d) [10 marks] If we now include the background flow $\bar{u}$, stationary wave solutions with zero phase velocity can be found. Show that there are two stationary wave solutions for a given k, and determine the qualitative wave paths in response to a disturbance in mid-latitudes. What might be the cause of such a disturbance? Hence, discuss how the dominant wavelengths of Rossby wave activity vary with latitude.