A15274W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

GEOPHYSICAL FLUID DYNAMICS

Trinity Term 2020

THURSDAY, 4TH JUNE 2020, 09:30 am

You should submit answers to both of the two questions.

You have **3 hours** *to complete the paper and upload your answer file.*
You are permitted to use the following material(s):
Calculator
Formula Sheet
The use of computer algebra packages is **not** *allowed*

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

1. Consider basis vectors $\mathbf{e}_i$ that are fixed in an inertial reference frame F, and corresponding basis vectors $\mathbf{e}'_i$ fixed in a reference frame F' that rotates with angular velocity $\boldsymbol{\Omega}$ with respect to F. Define the components A_i and A'_i of a vector $\mathbf{A}$ by $\mathbf{A} = \sum_{i=1}^{3} \mathbf{e}_i A_i = \sum_{i=1}^{3} \mathbf{e}'_i A'_i$.

   (a) [6 marks] If the rate of change in the inertial frame F is $(d\mathbf{A}/dt)_I = \sum_{i=1}^{n} \mathbf{e}_i (dA_i/dt)$ for time t, and the rate of change in the rotating frame F' is $(d\mathbf{A}/dt)_R = \sum_{i=1}^{n} \mathbf{e}'_i (dA'_i/dt)$, show that

$$\left(\frac{d\mathbf{A}}{dt}\right)_I = \left(\frac{d\mathbf{A}}{dt}\right)_R + \boldsymbol{\Omega} \times \mathbf{A}.$$

   Starting from the Navier-Stokes equations in an inertial reference frame F, show that

$$\frac{D\mathbf{u}}{Dt} + 2\boldsymbol{\Omega} \times \mathbf{u} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{r}) = -\frac{1}{\rho}\boldsymbol{\nabla} p + \frac{\mathbf{F}}{\rho},$$

   when viewed in a reference frame F' that rotates with constant angular velocity Ω. Here $\mathbf{u}$ is the fluid velocity, ρ the density, p the pressure, $\mathbf{F}$ represents applied body forces and viscous stresses, and $D/Dt = \partial/\partial t + \mathbf{u} \cdot \boldsymbol{\nabla}$ is the Lagrangian derivative. Identify the effective Coriolis and centrifugal forces.

   (b) [5 marks] Now consider fluid flow in a reference frame F' that rotates with time-dependent angular velocity $\boldsymbol{\Omega} = \Omega(t)\hat{\mathbf{z}}$ relative to the frame F, for some function $\Omega(t)$ and $\hat{\mathbf{z}}$ a unit vector. Derive the corresponding momentum equation viewed in the frame F'. If the fluid density is approximately constant, and Rossby number is small, explain why the equations of motion can be simplified into the form

$$\frac{\partial \mathbf{u}}{\partial t} + 2\boldsymbol{\Omega} \times \mathbf{u} = -\frac{d\boldsymbol{\Omega}}{dt} \times \mathbf{r} - \frac{1}{\rho}\boldsymbol{\nabla}\tilde{p} + \frac{\mathbf{F}}{\rho}, \quad \nabla \cdot \mathbf{u} = 0 \tag{1}$$

   for a redefined effective pressure $\tilde{p}$.

   (c) [8 marks] It has been hypothesised that the icy moon Enceladus has an interior liquid ocean, trapped between an inner rocky core and an outer solid ice shell. The gravitational interactions with other planetary bodies cause the core of Enceladus to librate with $\boldsymbol{\Omega} = \Omega_0(1 + \epsilon \cos \omega t)\hat{\mathbf{z}}$, where Ω_0, ϵ and ω are all constant. We approximate the ocean as a thin spherical shell of fluid occupying $a < r < b$, for constants a and b where r is the distance from the centre of Enceladus. Using spherical co-ordinates with pole aligned with the rotation axis, we consider a simple model of zonally-symmetric ocean flow with all variables independent of the azimuthal angle ϕ, $\mathbf{u} = u_\theta \hat{\boldsymbol{\theta}} + u_\phi \hat{\boldsymbol{\phi}}$, and radial fluid motion neglected ($u_r = 0$). The body force is dominated by frictional drag on the rocky core and ice shell, which can be parameterised as $\mathbf{F} = -\alpha\rho\mathbf{u}$ for constant α.

   Using mass conservation, show that $u_\theta = 0$. Hence determine a solution for the long-time evolution of the azimuthal velocity component u_ϕ in terms of t, θ, and the constant parameters. Explain what role the Coriolis force plays in the dynamics.

   (d) [6 marks] Starting from the equations (1), derive an equation for the rate of change of the domain-integrated kinetic energy in the form,

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_V \frac{1}{2}\rho\mathbf{u} \cdot \mathbf{u}\, dV = \text{Sources} - \text{Sinks},$$

   where V is the region $a < r < b$. Identify expressions for the main energy sources and sinks. What role might this energy budget play in the maintenance of the liquid ocean on Enceladus?

2. (a) [8 marks] Consider an inviscid shallow ocean of approximately constant density ρ on a β-plane, occupying $-h_0 < z < \eta(x, y, t)$ over a flat bed with h_0 constant. The overlying atmosphere imposes a constant pressure $p = p_{atm}$ at $z = \eta$. What are the boundary conditions on the fluid velocity at $z = -h_0$ and $z = \eta$?

If the layer thickness h is much smaller than the lengthscales of horizontal variation, use a scaling argument to argue why the vertical component of the inviscid Navier-Stokes equation (i.e. Euler momentum equation) simplifies to a hydrostatic balance. Hence derive the *shallow water equations*

$$\frac{D_H \mathbf{u}_H}{Dt} + f\hat{\mathbf{z}} \times \mathbf{u}_H = -g\boldsymbol{\nabla}_H \eta, \qquad \frac{\partial h}{\partial t} + \boldsymbol{\nabla}_H \cdot (h\mathbf{u}_H) = 0,$$

where the horizontal velocity $\mathbf{u}_H(x, y, t) = (u, v, 0)$, horizontal gradient $\boldsymbol{\nabla}_H = (\partial/\partial x, \partial/\partial y, 0)$, $f(y)$ is the Coriolis parameter on a β-plane, g gravitational acceleration and $D_H/Dt = \partial/\partial t + \mathbf{u}_H \cdot \boldsymbol{\nabla}_H$.

(b) [4 marks] Explain why $f(y) \approx \beta y$ for flows near to the equator. For such a near equatorial flow, linearise the shallow water equations for small amplitude disturbances about a state of rest, with $\eta \ll h_0$.

(c) [6 marks] The above linearised shallow water equations admit plane wave solutions of the form $(u, v, \eta) = \mathrm{Re}\,\{[\hat{u}(y), \hat{v}(y), \hat{\eta}(y)] \exp(ikx - i\omega t)\}$, where $\hat{v}$ satisfies

$$\frac{d^2\hat{v}}{dy^2} + \hat{v}\left(\frac{\omega^2}{gh_0} - k^2 - \frac{\beta k}{\omega} - \frac{\beta^2}{gh_0}y^2\right) = 0.$$

You do not need to derive this equation.

You are also given that the Hermite functions $\psi_n(\xi) = H_n(\xi)\exp(-\xi^2/2)$ satisfy

$$\psi_n'' + (2n + 1 - \xi^2)\psi_n = 0,$$

where $H_n(\xi)$ is the Hermite polynomial of order n, as defined by $H_0 = 1$, $H_1 = \xi$, and $H_{n+1}(\xi) = 2\xi H_n(\xi) - 2nH_{n-1}(\xi)$ for $n \geqslant 2$.

Show that the dispersion relation yields a quantised set of frequencies satisfying

$$\left(\frac{\omega^2}{c^2} - k^2 - \frac{\beta k}{\omega}\right)\frac{c}{\beta} = 2n + 1, \tag{2}$$

for some c to be determined. Determine the characteristic lengthscale for the meridional extent of these waves. Which of these modes permit cross equatorial flow?

(d) [7 marks] Now consider the solution branch of the dispersion relation with low frequency waves satisfying $|\omega| \ll c|k|$, and $n = 0$. By appropriately approximating equation (2), determine the leading order behaviour of ω as a function of k and constant parameters. In which direction does the wave phase propagate, and in which direction does the energy propagate?

By considering conserved quantities for shallow water flows and a suitable sketch, describe the physical restoring mechanism responsible for the propagation of such waves.