A15274W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

GEOPHYSICAL FLUID DYNAMICS

Trinity Term 2018

THURSDAY, 19TH APRIL 2018, 2:30pm to 4:30pm

You should submit answers to two of the three questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. (a) [5 marks] Consider a time-dependent flow over a flat ocean bottom at $z = 0$, near the Earth's North pole. Far above the bottom boundary, the velocity has zonal and meridional components $u_0 = \Re\left[U e^{i\omega t}\right]$ and $v_0 = 0$ respectively, where U and ω are constant. The ocean has constant density ρ, constant Coriolis parameter f, and a constant enhanced eddy viscosity ν.

   Starting from the incompressible Navier Stokes equations in a rotating reference frame $\rho\left(D\mathbf{u}/Dt + f\mathbf{e}_z \times \mathbf{u}\right) = -\nabla p - \rho g \mathbf{e}_z + \rho\nu\nabla^2\mathbf{u}$ and $\nabla\cdot\mathbf{u} = 0$, outline the key approximations required to derive the system

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$
$$\frac{\partial u}{\partial t} - fv = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu\frac{\partial^2 u}{\partial z^2},$$
$$\frac{\partial v}{\partial t} + fu = -\frac{1}{\rho}\frac{\partial p}{\partial y} + \nu\frac{\partial^2 v}{\partial z^2},$$
$$0 = -\frac{1}{\rho}\frac{\partial p}{\partial z} - g,$$

   for time-dependent flow in an Ekman layer.

   (b) [12 marks] Assuming that viscous effects are negligible far above the boundary, determine the simplified dynamical equations for the velocity components $u = u_0$, $v = v_0$ and the pressure $p = p_0$ in the interior flow.

   Decompose the total velocity as $u = u_0 + u_1$, $v = v_0 + v_1$ and $p = p_0 + p_1$, where u_1, v_1 and p_1 describe corrections in the Ekman layer. What are the boundary conditions on u_1 and v_1?

   Eliminating the pressure p, find a pair of differential equations for the velocity perturbations u_1 and v_1. Solve this system by considering a combination of the form $\zeta = \hat{u}_1 + i\hat{v}_1$ for complex $\hat{u_1}$ and $\hat{v_1}$, or otherwise. Hence deduce that

$$u = U\cos\omega t - U\exp\left[-\sqrt{\frac{f+\omega}{2\nu}}z\right]\cos\left[\omega t - \sqrt{\frac{f+\omega}{2\nu}}z\right],$$
$$v = -U\exp\left[-\sqrt{\frac{f+\omega}{2\nu}}z\right]\sin\left[\omega t - \sqrt{\frac{f+\omega}{2\nu}}z\right].$$

   (c) [8 marks] Calculate the basal shear stress $\tau = \rho\left(\nu\partial u/\partial z, \nu\partial v/\partial z, 0\right)$ at $z = 0$, and the Ekman transport

$$(U_{Ek}, V_{Ek}) = \int_0^\infty (u_1, v_1)\, dz.$$

   Compare the orientations of the drag and Ekman transport, and provide a physical explanation for their relative orientations.

2. (a) [7 marks] The shallow water equations for a layer of fluid with thickness $h(x, y, t)$, free-surface elevation η and horizontal velocity $\mathbf{u} = (u, v, 0)$ are

$$\frac{\partial h}{\partial t} + \nabla \cdot (\mathbf{u}h) = 0, \qquad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\,\mathbf{u} + f\mathbf{e}_z \times \mathbf{u} = -g\nabla\eta,$$

where g is the gravitational acceleration.

Explain the condition required to make the f-plane approximation.

Consider small amplitude linearised disturbances about a background state of rest and uniform water depth h_0 on an f-plane. Show that the free-surface elevation satisfies

$$\frac{\partial}{\partial t}\left[\frac{\partial^2 \eta}{\partial t^2} + f^2\eta - gh_0\nabla_H^2\eta\right] = 0,$$

where the horizontal Laplacian operator is $\nabla_H^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

(b) [4 marks] Consider waves of the form $\eta = \Re\left[\mathrm{e}^{ikx - i\omega t} H(y)\right]$ propagating in a long fjord on an f-plane, corresponding to a long channel of width $0 < y < L$. If the boundaries of the fjord at $y = 0$ and $y = L$ are impermeable, show that η must satisfy the boundary conditions

$$\eta = -\frac{\omega}{kf}\frac{\partial \eta}{\partial y}, \qquad \text{at} \quad y = 0,\ L.$$

(c) [9 marks] If the frequency is real $\omega > 0$, show that ω satisfies either

$$\omega^2 = \omega_n^2 = f^2 + gh_0\left(k^2 + \frac{n^2\pi^2}{L^2}\right), \qquad n = 1,\ 2,\ \ldots, \tag{1}$$

or

$$\omega^2 = \varpi^2 = f^2 + gh_0\left(k^2 + l^2\right), \qquad \text{with} \quad l^2 = -\frac{k^2 f^2}{\varpi^2}. \tag{2}$$

Determine the possible values of ϖ. If k and ϖ are both real, explain how the meridional structure of the waves differs for frequencies ω_n compared to waves with frequency ϖ.

(d) [5 marks] Two fjords of identical width $L = 10\,\mathrm{km}$ and depth $h_0 = 1\,\mathrm{km}$ are located in Norway at latitude 60°N, and in Patagonia at latitude 30°S. Discuss how each fjord will respond to forcing with a spectrum of many different wavelengths and angular frequency $\omega = 8 \times 10^{-5}\,\mathrm{rad\,s^{-1}}$, given a planetary rotation rate $\Omega = 7 \times 10^{-5}\,\mathrm{rad\,s^{-1}}$.

3. (a) [5 marks] The quasigeostrophic approximation for geophysical flows leads to the equations

$$\left[\frac{\partial}{\partial t} + \mathbf{u}_g \cdot \nabla\right] Q = 0, \qquad Q = \nabla_H^2 \psi + \frac{f^2}{N^2}\frac{\partial^2 \psi}{\partial z^2} + f, \tag{3}$$

where

$$\mathbf{u}_g = \left(-\frac{\partial \psi}{\partial y}, \frac{\partial \psi}{\partial x}, 0\right), \quad b = f\frac{\partial \psi}{\partial z}, \quad w = -\frac{1}{N^2}\left[\frac{\partial}{\partial t} + \mathbf{u}_g \cdot \nabla\right] b, \quad \nabla_H^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

and we have assumed that $f > 0$, N and the gravitational acceleration g are all constant. Provide a physical interpretation of each of the equations (3) including each of the terms in Q, and define the quantity N^2.

(b) [8 marks] Eady's model of instability considers the development of perturbations ψ' to a vertically sheared background flow, with $\psi = -\Lambda yz + \psi'$ over a region of depth $0 < z < H$, for constant $\Lambda > 0$. If $w = 0$ at the upper and lower boundaries $z = 0, H$, deduce the corresponding boundary conditions on ψ'.
Consider two-dimensional perturbations with small amplitude of the form $\psi' = \Re\left[\Phi(z)\mathrm{e}^{ikx-ikct}\right]$. Show that $\Phi(z)$ takes the form

$$\Phi(z) = A \sinh \nu z + B \cosh \nu z$$

where c^2 satisfies

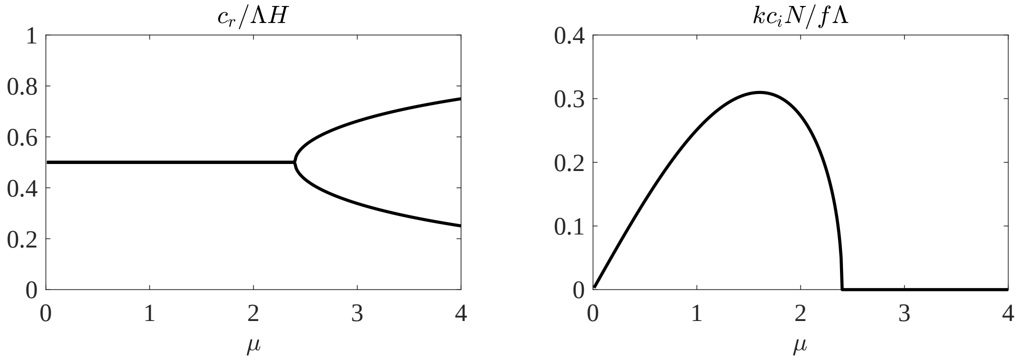
$$c^2 - \Lambda H c + \Lambda^2 H^2 \left(\frac{\coth \mu}{\mu} - \frac{1}{\mu^2}\right) = 0,$$

for some ν and μ to be determined, and where A and B are unspecified integration constants.

(c) [6 marks] The solution for c is

$$c = \frac{\Lambda H}{2} \pm \frac{\Lambda H}{2}\left[\left(\frac{\mu}{2} - \coth\frac{\mu}{2}\right)\left(\frac{\mu}{2} - \tanh\frac{\mu}{2}\right)\right]^{1/2},$$

and the figure below plots $c_r/\Lambda H$ and $kc_iN/f\Lambda$ as a function of μ, where $c_r = \Re[c]$ and $c_i = \Im[c]$. Determine the conditions required for instability in terms of k, H, f and N. If the initial state is perturbed by noise with equal amplitude at all wavenumbers, discuss what you expect to observe over subsequent times and determine an expression for the growth rate of the resulting instability.



(d) [6 marks] Modes of the above form result in an eddy flux

$$\overline{v'b'} = \frac{kc_i\Lambda H f}{2\,|c|}\exp\left(2kc_it\right),$$

where the overbar operator denotes a zonal average. The initial density stratification before the onset of instability is $\rho = \rho_0 \left[1 - N^2 z/g + f_0 \Lambda y/g\right]$.

With the aid of suitable sketches, explain how surfaces of constant density will evolve as the instability develops, explaining why this is consistent with the sign of $\overline{v'b'}$.

Discuss whether this instability is more consistent with a barotropic or baroclinic instability, explaining your answer.