Honour School of Mathematical and Theoretical Physics Part C

GEOPHYSICAL FLUID DYNAMICS

Trinity Term 2017

THURSDAY, 20TH APRIL 2017, 2.30pm to 4.30pm

You should submit answers to two of the three questions.

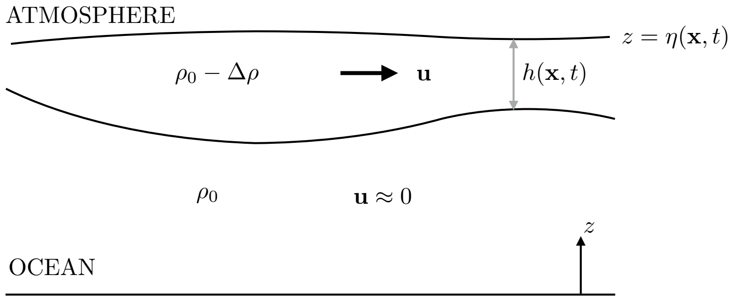
You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

A formula sheet will be provided.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. (a) [6 marks] Consider a shallow layer of fluid with constant density $\rho_0 - \Delta\rho$ lying atop a deep ocean of constant density ρ_0 on an f-plane, as illustrated in the figure below. The free surface is at height $z = \eta(x,t)$, f is the Coriolis parameter and there is gravitational acceleration g.



The evolution of the horizontal velocity $\mathbf{u} = (u, v)$ and thickness h of the upper layer are determined by the *Boussinesq reduced-gravity equations*, which take the form

$$\frac{\partial h}{\partial t} + \nabla_H \cdot (\mathbf{u}h) = 0, \qquad \frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla_H)\,\mathbf{u} + f\mathbf{e}_z \times \mathbf{u} = -\alpha \nabla_H h,$$

where $\nabla_H = (\partial/\partial x, \partial/\partial y)$ is the horizontal gradient. By considering the vertical momentum balance and assuming that the deep layer is at rest, determine the constant α.

(b) [7 marks] Derive the vorticity equation, and hence show that

$$\left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla_H\right) Q = 0, \qquad \text{where} \quad Q = \frac{f\mathbf{e}_z + \nabla \times \mathbf{u}}{h}.$$

Provide a physical interpretation of this result and the relation to conservation of angular momentum.

(c) [2 marks] Write down the appropriate form of the reduced-gravity equations for a steady flow in geostrophic balance.

(d) [10 marks] An oil spill generates a mixture of density $\rho_0 - \Delta\rho$ in an initially cylindrical surface layer with uniform depth h_0 and radius R_0. The surrounding ocean has density ρ_0. The oil mixture adjusts to a steady, axisymmetric, geostrophically-balanced state with depth $h(r)$, where r is the horizontal distance from the centre of the spill. Let $L = \sqrt{\alpha h_0}/f$. Determine $h(r)$, assuming that $h(R) = 0$ at the edge of the spill for some unknown radius R. By enforcing mass conservation, show that R satisfies

$$\int_0^R \left[1 - \frac{I_0(\gamma r)}{I_0(\gamma R)}\right] r\, dr = \frac{R_0^2}{2},$$

where $I_0(x)$ is the zeroth-order modified Bessel function, and determine γ as a function of L. Provide a physical interpretation for the value of γ.

Comment briefly on whether the final geostrophically balanced state would change if the initial condition contained an identical volume of fluid, but distributed as (i) a cuboid of uniform depth h_0, and (ii) half a sphere with $h_0(r) = \sqrt{a^2 - r^2}$ for appropriate a.

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[*You may quote the following results:*

In cylindrical co-ordinates (r, ϕ, z), *the curl of* $\mathbf{F} = F_r\mathbf{e}_r + F_\phi\mathbf{e}_\phi + F_z\mathbf{e}_z$ *is*

$$\nabla \times \mathbf{F} = \mathbf{e}_r \left[\frac{1}{r}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right] + \mathbf{e}_\phi \left[\frac{\partial F_r}{\partial z} - \frac{\partial F_z}{\partial r}\right] + \mathbf{e}_z \left[\frac{1}{r}\frac{\partial (rF_\phi)}{\partial r} - \frac{1}{r}\frac{\partial F_r}{\partial \phi}\right]$$

The zeroth-order modified Bessel equation

$$\frac{\partial^2 y}{\partial x^2} + \frac{1}{x}\frac{\partial y}{\partial x} - y = 0$$

has general solution

$$y = \alpha I_0(x) + \beta K_0(x)$$

where $I_0(0) = 1$, $K_0 \to \infty$ *as* $x \to 0$, *and* $I_0 \to \infty$, $K_0 \to 0$ *as* $x \to \infty$.]

2. (a) [7 marks] Consider a shallow atmosphere with density ρ on a rocky planet of radius a rotating with angular velocity $\Omega \mathbf{e}_z$. Determine an expression for the angular momentum $m(\phi)$ of a parcel of fluid at latitude ϕ, moving with zonal wind speed $u(\phi)$ relative to the planet surface. If a parcel of fluid starts from rest at the equator ($\phi = 0$) and then moves poleward conserving its angular momentum, deduce that

$$u(\phi) = \Omega a \frac{\sin^2 \phi}{\cos \phi}.$$

Sketch $u(\phi)$, and estimate the magnitude of the zonal wind speed at latitude 45°N for the Earth, which has radius $a \sim 6400$km and rotational period of 1 day. Explain why this model breaks down at sufficiently high latitudes.

(b) [4 marks] Assume that the fluid is Boussinesq, and the equation of state linking the density ρ to the potential temperature θ of the fluid can be approximated by $\rho \approx \rho_0(1 - \theta/\theta_0)$, for constant θ_0 and ρ_0. If the planetary scale zonal flow is in geostrophic and hydrostatic balance, derive the *thermal wind relation*

$$2\Omega \sin \phi \frac{\partial u}{\partial z} = -\frac{g}{a} \frac{\partial}{\partial \phi} \left(\frac{\theta}{\theta_0} \right),$$

where $z = r - a$ is the height above the planetary surface.

(c) [6 marks] The *Held-Hou model* describes an overturning circulation with an upper flow that conserves angular momentum with approximate geostrophic and hydrostatic balance, and a lower frictionally dominated return flow with negligible zonal velocity. Using a small angle approximation with $\phi \ll 1$, determine the vertically-averaged temperature as a function of latitude,

$$\overline{\theta}(\phi) = \frac{1}{H} \int_0^H \theta \, dz,$$

where H is the height of the top of the overturning cell. Without further detailed calculation, provide a brief physical explanation of how the extent of this overturning Hadley cell is determined in the Held-Hou model.

(d) [8 marks] The zonal-mean flow in the midlatitudes of the Earth (poleward of the Hadley cell) is strongly influenced by turbulent eddies and waves. In the *Transformed Eulerian Mean framework*, the quasigeostrophic zonally-averaged zonal-momentum equation and eddy fluxes of potential vorticity are given by

$$\frac{\partial \overline{u}}{\partial t} = f_0 \overline{v}^* + \overline{v'q'} + \overline{D}, \qquad \overline{v'q'} = -\frac{\partial}{\partial y} \left(\overline{u'v'} \right) + \frac{\partial}{\partial z} \left(\frac{f_0}{N^2} \overline{v'b'} \right),$$

where overlines now denote a zonal average, $\overline{D}$ describes dissipation and $\overline{v}^*$ is the residual mean meridional velocity. Provide a physical interpretation for the terms in this zonal momentum equation.

Barotropic Rossby waves are generated in the mid-latitude jet stream, and propagate with disturbance streamfunctions

$$\psi_\pm = \Re \left[A_\pm e^{i(kx + l_\pm y - \omega t)} \right], \qquad \text{with} \quad \omega = Uk - \frac{\beta k}{k^2 + l_\pm^2},$$

before breaking at a different latitude, as illustrated in the figure on the next page. The jet stream initially has constant mean speed U, the zonal wavenumber $k > 0$ and the meridional wavenumbers $l_\pm$ have different sign on each side of the jet.

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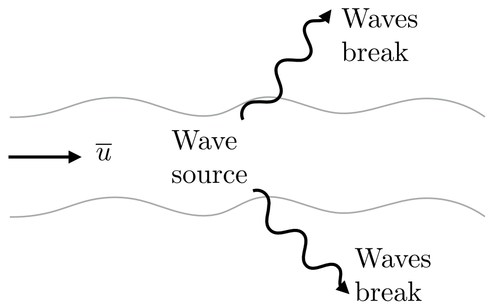

Calculate the group velocity and eddy-momentum flux $\overline{u'v'}$ of the waves. Hence explain how wave evolution contributes to the momentum budget of the jet stream. Do such waves accelerate, or decelerate the jet stream?

3. (a) [6 marks] The Boussinesq equations for motion of a linearly stratified inviscid fluid take the form

$$\rho_0 \left[\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla) \mathbf{u} \right] = -\nabla p - \rho' g \mathbf{e}_z, \qquad \nabla \cdot \mathbf{u} = 0, \qquad \frac{\partial \rho'}{\partial t} + (\mathbf{u} \cdot \nabla) \rho' - \rho_0 w \frac{N^2}{g} = 0.$$

Provide a physical interpretation for each of these equations, explaining the meaning of ρ_0, and ρ'. What does the quantity N represent, and how is it related to the stratification? What condition is required on the characteristic scales of the flow to justify the neglect of Coriolis effects?

(b) [9 marks] Consider two dimensional disturbances in the $(x, 0, z)$ plane. Linearise the equations of motion about a state of rest. Show that the dispersion relation for plane wave solutions with angular frequency ω and wavevector $\mathbf{k} = (k, 0, m)$ is

$$\omega^2 = \frac{N^2 k^2}{k^2 + m^2}.$$

Calculate the phase speeds and group velocity of the wave. Sketch the lines of constant phase in the region $z > 0$ for a plane wave generated by a localised source at $z = 0$, clearly indicating the directions of phase propagation and energy propagation. How would the dispersion relation differ if the flow was approximately hydrostatic?

(c) [4 marks] Now consider an atmosphere consisting of two layers of fluid with different density gradients, with $N = N_1$ for $0 < z < H$, and $N = N_2$ for $H < z < \infty$. A disturbance at ground level generates a flow with $w = w_0 \cos(kx - \omega t)$ at $z = 0$. You may assume that the vertical wavenumber satisfies $mH \gg 1$. State linearised boundary conditions that apply at the interface $z = H$ between the two layers. If the wave solution takes the form

$$(u, w, p, \rho') = \Re \left[(u_1, w_1, p_1, \rho_1') f(x, z, t) \right],$$

determine $f(x, z, t)$ in each layer $0 < z < H$ and $H < z < \infty$, to within a complex multiplicative factor. *There is no need to find (u_1, w_1, p_1, ρ_1') in each layer.*

(d) [6 marks] Given an imposed horizontal wavenumber k, determine the condition required on ω, N_1 and N_2 for the wave energy to reach and propagate freely through the upper layer $H < z < \infty$. Determine the corresponding condition on ω for an internal wave to propagate through M successive layers with

$$N = N_j \qquad \text{for} \quad (j-1)H < z < jH, \qquad j = 1, 2 \ldots, M.$$

Gravity waves are generated at ground level, and observed to propagate upward to the stratosphere with time periods τ in the range $6\text{s} \leqslant \tau \leqslant 600\text{s}$. What can be deduced about the stratification in the underlying troposphere?