A15274W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

GEOPHYSICAL FLUID DYNAMICS

Trinity Term 2016

WEDNESDAY, 20 APRIL 2016, 14.30 to 16.30

You should submit answers to two of the three questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. (a) [7 marks] Write down the *Navier Stokes equations* for a uniform incompressible fluid of constant density ρ_0 in a reference frame rotating with constant angular velocity $\mathbf{\Omega}$. Define the *Rossby number* Ro and *Ekman number* Ek. For Ro $\ll 1$ and Ek $\ll 1$ explain why the velocity $\mathbf{u}$, pressure p and density ρ_0 approximately satisfy

$$2\mathbf{\Omega} \times \mathbf{u} = -\frac{1}{\rho_0}\nabla p + \mathbf{g},$$

where $\mathbf{g}$ should be identified. If the curl of $\mathbf{g}$ is zero, prove *the Taylor-Proudman theorem.*

[*You may find the vector identity*

$$\nabla \times (\mathbf{A} \times \mathbf{B}) = \mathbf{A}\,(\nabla \cdot \mathbf{B}) - \mathbf{B}\,(\nabla \cdot \mathbf{A}) + (\mathbf{B} \cdot \nabla)\,\mathbf{A} - (\mathbf{A} \cdot \nabla)\,\mathbf{B}.$$

to be useful.]

(b) [6 marks] A new approximately spherical exoplanet is discovered that rotates with a time period of 16 hours and is hypothesized to consist entirely of incompressible fluid of constant density, with mean radius $a = 10^7\,\mathrm{m}$ and kinematic viscosity $\nu = 10^{-6}\,\mathrm{m^2\,s^{-1}}$. At the planet's surface, the time mean winds are observed to vary with latitude θ as $\mathbf{u} = U_0 \cos 5\theta \mathbf{e}_\phi$ where $U_0 = 100\,\mathrm{m\,s^{-1}}$ and ϕ is the azimuthal angle. If $\mathbf{g}$ is dynamically irrelevant, explain how the time-mean flow varies in the interior $r < a$ of the spherical planet, justifying any approximations used. Sketch the velocity field in the equatorial plane $\theta = 0$.

(c) [9 marks] The conservation of potential vorticity results in the relation

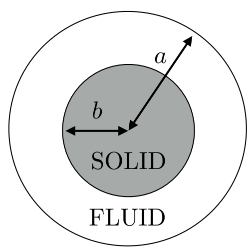
$$\frac{D}{Dt}\left[\frac{2\Omega + \zeta}{h}\right] = 0,$$

where $h(R,\phi)$ is the thickness of a fluid column that spans the full depth of the planet at a perpendicular distance R from the rotation axis, and ζ is the component of relative vorticity parallel to the rotation axis. Provide a physical interpretation for this relation.

A small displacement of a fluid column away from the rotation axis triggers a topographic Rossby wave with surface velocity

$$\mathbf{u} = \mathbf{u}_0 + \Re\left[\mathbf{u}_1(\theta)\mathrm{e}^{i\omega t - ikR_0\phi}\right],$$

near to the latitude $\theta_0 = 74^\circ$, where $\mathbf{u}_0(\theta,\phi)$ describes the time-mean background flow, and $R_0 = a\cos\theta_0$. By considering the potential vorticity, sketch a diagram indicating how the sign of the perturbation to the relative vorticity evolves along the wave. Hence explain the physical mechanism that allows the wave to propagate, and how flow is induced. Without further detailed calculation, determine whether the wave propagates forwards or backwards relative to the mean flow. How would your answer differ if the planet had a solid spherical inner core of radius b, as sketched in the figure below, with $b > a/3$?



(d) [3 marks] Discuss briefly how the breaking of atmospheric waves can contribute to the momentum budget of the time-mean flow.

2. The quasigeostrophic dynamics of a continuously stratified fluid with constant buoyancy frequency N on a β-plane can be described via the system

$$\left(\frac{\partial}{\partial t} + u_g\frac{\partial}{\partial x} + v_g\frac{\partial}{\partial y}\right) Q = 0,$$

where

$$Q = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial}{\partial z}\left(\frac{f_0^2}{N^2}\frac{\partial\psi}{\partial z}\right) + f_0 + \beta y.$$

(a) [5 marks] State two key assumptions that are required in order to derive these equations starting from the Navier-Stokes equations on a β-plane. What does the quantity Q represent? Express each of u_g, v_g and the dynamic pressure p in terms of ψ.

(b) [7 marks] Consider a flow with $\psi = -Uy + \psi'$ corresponding to a disturbance ψ' to a background wind of constant speed U. Determine linearised dynamical equations for small disturbances with $|\psi'| \ll Uy$. By looking for plane wave solutions of the form

$$\psi' = \Re\left[B e^{i(kx+ly+mz-\omega t)}\right],$$

for $k, l, m, \omega, \in \mathbb{C}$, show that the dispersion relation satisfies

$$\omega = Uk - \frac{\beta k}{k^2 + l^2 + f_0^2 m^2/N^2}.$$

What do the solutions correspond to when $\beta = 0$? Provide a physical interpretation of this result.

(c) [5 marks] Wind blows over rough topography, and generates a disturbance

$$\psi' = A\cos kx, \qquad \text{at} \quad z = 0,$$

for given A and k that are real and constant. Assuming that $\beta > 0$, determine the condition required for the existence of stationary waves, where the wave crests and troughs do not move relative to the ground. By considering the vertical structure of the waves, show that stationary waves can only propagate vertically for wind speeds in the range

$$U_1 < U < U_2,$$

and determine the critical values U_1 and U_2.

(d) [8 marks] Now consider stationary waves in the stratosphere induced by a pressure perturbation at the tropopause of amplitude 300Pa, with a zonal wavelength of 10^4 km where $f_0 \approx 10^{-4}\,\mathrm{s}^{-1}$, $\beta \approx 10^{-11}\,\mathrm{m}^{-1}\mathrm{s}^{-1}$ and $N^2 = 4\times10^{-4}\,\mathrm{s}^{-2}$. For each of the two cases $U = -10\,\mathrm{m\,s}^{-1}$ and $U = +10\,\mathrm{m\,s}^{-1}$, estimate the characteristic vertical scales of the flow, and sketch the variation of the dynamic pressure p' with height above the tropopause. Discuss the competing effects of rotation and stratification on the vertical structure of this quasigeostrophic flow. Explain how the flow would differ if the background vertical density gradient was four times larger.

3. Consider an ocean of constant depth H and constant density ρ on a β-plane. The circulation can be approximated by

$$-\rho f v = -\frac{\partial p}{\partial x} + \frac{\partial \tau^{(x)}}{\partial z}, \qquad \rho f u = -\frac{\partial p}{\partial y} + \frac{\partial \tau^{(y)}}{\partial z}, \qquad 0 = -\frac{\partial p}{\partial z} - \rho g, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0,$$

where $f = f_0 + \beta y$. The turbulent stress components $\tau^{(x)}$ and $\tau^{(y)}$ are non-zero only in surface and bottom boundary layers of constant thickness δ, and decay to zero in the ocean interior for $-H + \delta < z < -\delta$.

(a) [8 marks] Consider flow driven by a turbulent surface wind stress $\boldsymbol{\tau}_s = (\tau_s^{(x)}, \tau_s^{(y)}, 0)$, and with constant atmospheric pressure $p = p_{atm}$ at $z = 0$. Determine the pressure $p(x, y, z)$, and find the *depth-integrated Ekman transport*

$$\mathbf{U}_{Ek} = \int_{-\delta}^{0} (u, v, 0)\, \mathrm{d}z,$$

in the surface mixed layer. Using a suitable sketch, describe the force balance and direction of the depth-integrated flow in a slab of the mixed layer in the Northern hemisphere that experiences constant wind stress $\boldsymbol{\tau}_s = (\tau_0, 0, 0)$, where $\tau_0 > 0$.
A uniform northward wind blows parallel to a coastline at $x = 0$. Explain what direction of vertical transport is induced for $x < 0$ near to the coastline.

(b) [7 marks] Now consider the corresponding flow in the ocean interior, accounting for both surface wind stress $\boldsymbol{\tau}_s = (\tau_s^{(x)}, \tau_s^{(y)}, 0)$ at $z = 0$ and basal friction based on the linear drag law $\boldsymbol{\tau} = \rho r(U, V, 0)$ at $z = -H$ where r is constant and

$$\mathbf{U} = (U, V, 0) = \int_{-H}^{0} (u, v, 0)\, \mathrm{d}z = \left(-\frac{\partial \Psi}{\partial y}, \frac{\partial \Psi}{\partial x}, 0\right),$$

can be expressed in terms of a two-dimensional depth-integrated streamfunction $\Psi(x, y)$. For flow in a closed square basin $\mathcal{B} = \{(x, y) \colon 0 < x < L,\, 0 < y < L\}$, explain why we can set $\Psi = 0$ on the boundary, and show that Ψ satisfies

$$\beta \frac{\partial \Psi}{\partial x} = \frac{1}{\rho}\left[\frac{\partial \tau_s^{(y)}}{\partial x} - \frac{\partial \tau_s^{(x)}}{\partial y}\right] - r \nabla^2 \Psi. \tag{1}$$

Show further that

$$\int_{\mathcal{B}} \frac{1}{\rho}\left[\frac{\partial \tau_s^{(y)}}{\partial x} - \frac{\partial \tau_s^{(x)}}{\partial y}\right] \mathrm{d}A = \int_{\partial \mathcal{B}} r \mathbf{U} \cdot \mathrm{d}\mathbf{l}, \tag{2}$$

where $\partial \mathcal{B}$ is the boundary of the basin $\mathcal{B}$.

(c) [10 marks] In the limit of weak dissipation $r/\beta L \ll 1$, explain why $-r\nabla^2\Psi$ is a comparatively small term in (1). Assume that $-r\partial^2\Psi/\partial y^2$ can be neglected (as a regular perturbation), but $-r\partial^2\Psi/\partial x^2$ must be retained to satisfy the boundary conditions (and avoid a singular perturbation). After making this approximation, use separation of variables to find an approximate solution Ψ for a purely zonal wind stress $\boldsymbol{\tau}_s = (\tau_0 \cos \pi y/L, 0, 0)$. Sketch the streamlines when $f_0 > 0$, labelling key features of the flow. Identify the regions with sources and sinks of vorticity. Discuss the roles of zonal and meridional transport in maintaining the global vorticity balance (2).