Final Honour School of Mathematical and Theoretical Physics Part C and MSc Mathematical and Theoretical Physics

Galactic and Planetary Dynamics

The steps for each part of the miniproject are for your guidance; if you wish to take an alternative route to the desired goal, you are free to do so. But, if you follow the suggested route and find yourself unable to carry out any particular step, you may simply assume it so that you can continue with the miniproject, but should make this assumption clear in your presentation.

Please write or print on one side of the paper only.

Table 1 provides an instantaneous snapshot of the positions $\mathbf{r} = (x, y, z)$ and velocities $\mathbf{v} = (\dot{x}, \dot{y}, \dot{z})$ of the bodies within a certain flattened stellar system. The stars move in a common potential of the form

$$\Phi(r) = \frac{M}{\gamma} r^{\gamma}, \tag{1}$$

where r is the usual three-dimensional radius, and M and γ are unknown parameters that we want to constrain.

(i) Why is it reasonable to assume that the parameter γ lies within the range $-1 \leq \gamma \leq 2$?

(ii) By multiplying the Collisionless Boltzmann equation by appropriate functions of $\mathbf{x}$ and $\mathbf{v}$ and integrating, show that

$$M \simeq \sum_{n=1}^{N} \mathbf{v}_n^2 \bigg/ \sum_{n=1}^{N} r_n^{\gamma},$$

stating clearly any assumptions that you make.

(iii) Estimate M for the system in Table 1 for the cases $\gamma = -1$ and $\gamma = 2$.

(iv) Part (ii) above involves an explicit expression for the parameter M as a ratio of two functions that are independent of M. Do functions $A(\mathbf{x}, \mathbf{v}, M)$ and $B(\mathbf{x}, \mathbf{v}, M)$, both independent of γ, exist for which

$$\gamma \simeq \sum_{n=1}^{N} A(\mathbf{x}_n, \mathbf{v}_n, M) \bigg/ \sum_{n=1}^{N} B(\mathbf{x}_n, \mathbf{v}_n, M)?$$

Justify your answer.

An alternative approach is to view the problem in angle-action coordinates $(\boldsymbol{\theta}, \mathbf{J})$. Of course, the mapping from $(\mathbf{x}, \mathbf{v})$ to $(\boldsymbol{\theta}, \mathbf{J})$ depends on the unknown potential Φ, but there are various methods that use the resulting distribution of $(\boldsymbol{\theta}, \mathbf{J})$ to select the "best" such mapping and hence the best potential.

(v) How are the radial and azimuthal action coordinates J_r, J_ϕ defined for the potential (1)? Explaining the method you use, calculate numerical values of J_r for each of the stars in Table 1 for the cases $M = 1$ and (a) $\gamma = -1$, (b) $\gamma = 2$ and (c) $\gamma = -0.5$.

One such action-based method is that of Tremaine (2018, arxiv:1712.08794, hereafter T18). Consider how the values of the angle-action coordinates $\mathbf{z} = (\boldsymbol{\theta}, \mathbf{J})$ change under the flow

$$\frac{\mathrm{d}\bullet}{\mathrm{d}\lambda} = \mathcal{L}_\chi \bullet \equiv [\bullet, \chi]$$

generated by some function χ, where $\bullet$ stands for any function of $\mathbf{z}$. Along the flow we have that

$$\mathbf{z}(\lambda) = \exp[\lambda \mathcal{L}_\chi]\mathbf{z}(0).$$

Given a function $P(\mathbf{z})$, define P_λ through

$$P_\lambda(\mathbf{z}(\lambda)) = P(\mathbf{z}(0)). \tag{2}$$

(vi) What is the role of the flow parameter λ and the functions χ and P in the T18 method? Hence show that, under assumptions that you should identify, equation (2) reduces to $1 = \langle [\chi, P] \rangle_{\boldsymbol{\theta}}$, where $\langle \bullet \rangle_{\boldsymbol{\theta}}$ denotes an average over angle. Explain how the solution to this equation for P leads to an improved estimate of the potential parameters.
[Restrict your answer to the case in which the potential depends on just one unknown parameter.]

(vii) Suppose you were to implement the T18 method to estimate the parameters M and γ for the stellar system in Table 1. Identify the most significant problems that you would need to solve in order to achieve this, and – without detailed calculation – indicate how you would tackle them.

x	y	z	$\dot{x}$	$\dot{y}$	$\dot{z}$
0.324	0.091	0	−0.023	−4.628	0
−0.702	−0.169	0	1.725	−7.205	0
−0.983	−0.191	0	1.126	−6.187	0
1.104	−0.826	0	3.260	4.524	0
3.266	−3.888	0	2.076	1.904	0
−9.219	1.788	0	−0.496	−2.005	0
19.931	−2.555	0	0.172	1.357	0
24.323	−17.606	0	0.664	0.935	0

Table 1: Instantaneous snapshot of the positions $\mathbf{r} = (x, y, z)$ and velocities $\mathbf{v} = (\dot{x}, \dot{y}, \dot{z})$ of a two-dimensional stellar system.