Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

DISC ACCRETION IN ASTROPHYSICS: THEORY AND APPLICATIONS TAKE HOME EXAM

Trinity Term 2020

TUESDAY, 23RD JUNE 2020, 12 noon to THURSDAY, 25TH JUNE 2020, 12 noon

You should submit answers to all questions.

Answer all of the questions in Section A and one of the two essay questions from Section B. Answer booklets are provided for you to use but you may type your answers if you wish. Typed answers should be printed single-sided and the pages securely fastened together. You may refer to books and other sources when completing the exam but should not discuss the exam with anyone else.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

Section A: Accretion Disc Physics

Please answer **all questions** *in this section.*

1. Consider an axisymmetric accretion disc with surface density $\Sigma(R,t)$ orbiting around a central body with angular velocity $\Omega(R)$. By considering the disc as a set of interacting rings, or otherwise, show that

$$R\frac{\partial\Sigma}{\partial t} + \frac{\partial}{\partial R}(Ru\Sigma) = 0,$$

$$R\frac{\partial}{\partial t}(\Sigma R^2\Omega) + \frac{\partial}{\partial R}(R\,\Sigma u R^2\Omega) = \frac{1}{2\pi}\frac{\partial G_{\rm tot}}{\partial R},$$

where $u(R,t)$ is the radial velocity of the matter and $G_{\rm tot}(R,t)$ is the total torque exerted by the disc outside of radius R on the disc inside of that radius. [10]

2. Consider the accretion disc of question 1 in orbit around a central object of mass M, where radial pressure gradients are negligible and the mass of the accretion flow is negligible. Further, assume that this accretion disc is subject to the sum of an internal viscous torque, $G_\nu(R,t) = 2\pi R^3\nu\Sigma\Omega'$ (where ν is the kinematic viscosity and primes denote differentiation with respect to R) and an external torque, $G_{\rm m}(t)$, due to large-scale magnetic fields that connect it with the central object. Show that the evolution of the disc is governed by

$$\frac{\partial\Sigma}{\partial t} = \frac{3}{R}\frac{\partial}{\partial R}\left(R^{1/2}\frac{\partial}{\partial R}(\nu\Sigma R^{1/2})\right) - \frac{1}{\pi R(GM)^{1/2}}\frac{\partial}{\partial R}\left(R^{1/2}\frac{\partial G_{\rm m}}{\partial R}\right).$$

[10]

Consider the case where $G'_{\rm m} = \beta\delta(R - R_{\rm m})$, corresponding to all of the external torque being applied at a single radius $R = R_{\rm m} > R_*$ (where R_* is the innermost radius). Assume that the viscous torque vanishes at R_*. Show that, in a steady state, the local rate of viscous dissipation per unit surface area of the disc is given by

$$D_{\rm ss} = \frac{3GM\dot{M}}{8\pi R^3}\left(1 - \sqrt{\frac{R_*}{R}}\right) + \frac{3(GM)^{1/2}\beta}{8\pi R^{7/2}}\Theta(R - R_{\rm m}),$$

where $\Theta(x)$ is the Heaviside step function and $\dot{M}$ is the mass accretion rate onto the central object.

[10]

Sketch $D_{\rm ss}(R)$. Comment on its behaviour around $R = R_{\rm m}$ and at $R \gg R_{\rm m}$. [5]

[The viscous dissipation rate per unit surface area of the disc is $D(R) = \frac{1}{2}\nu\Sigma R^2\left(\frac{\partial\Omega}{\partial R}\right)^2$.]

3. Consider a Keplerian disc in which matter is injected at a constant rate $\dot{M}_0$ at radius R_0 around a central object of mass M. Within R_0 it spirals inwards under the action of the kinematic viscosity ν, accreting at radius R_* onto the central object. Outside R_0 a steady distribution of mass extends to very large radii and removes the angular momentum of the matter accreting within R_0. Explain why this implies that the mass-flow rate $\dot{M}$ as a function of radius R is 0 for $R > R_0$ and $\dot{M}_0$ for $R < R_0$. Use this and appropriate boundary/matching conditions at R_* and R_0 to show that in a steady state

$$\nu\Sigma = \frac{\dot{M}_0}{3\pi}\left[1 - \left(\frac{R_*}{R}\right)^{1/2}\right]$$

for $R < R_0$ and

$$\nu\Sigma = \frac{\dot{M}_0}{3\pi}\left[\left(\frac{R_0}{R}\right)^{1/2} - \left(\frac{R_*}{R}\right)^{1/2}\right]$$

for $R > R_0$, where Σ is the surface density. [15]

[Hints: express $\dot{M}$ in terms of the radial flow velocity u and use the expression for u in terms of the kinematic viscosity; adapted from Frank, King & Raine, Accretion Power in Astrophysics.]

4. Consider an *optically thin* layer on top of an *optically thick*, geometrically thin Keplerian accretion disc around a central object of mass M. In hydrostatic equilibrium, the energy balance (per unit mass) as a function of R in this atmospheric layer is given by

$$\frac{3}{2}\sqrt{\frac{GM}{R^3}}\alpha\,\frac{kT}{\mu m_{\rm H}} + \frac{\kappa F}{m_{\rm e}c^2}(E - 4kT) - \rho\Lambda(T) = 0,$$

where the first term is due to viscous heating (treated in the Shakura-Sunyaev model with viscosity parameter α), the second term describes Comptonization effects, and the third term radiative cooling (Λ is the so-called cooling function). In this equation, T and ρ are the temperature and density in the atmospheric layer, F is the total radiative flux generated locally in the disc, E is the flux-weighted mean energy of photons coming from the optically thick region, κ the opacity for electron scattering, μ the mean molecular weight and $m_{\rm e}$ and $m_{\rm H}$ the masses of the electron and the hydrogen atom, respectively.

Find $T(r)$ in the limit where $\rho \to 0$ (when the last term in the energy equation can be ignored). [4]

Using the expressions for a Shakura-Sunyaev disc (from the lectures) with an effective temperature $T_{\rm eff}(R)$ for which $E \simeq 3.83\,kT_{\rm eff}$, show that there is no solution for $T(r)$ unless

$$\frac{R^{3/2}}{f^4} < \frac{\kappa\mu m_{\rm H}\dot{M}\sqrt{GM}}{\pi m_{\rm e}c^2\alpha},$$

where $f^4 = 1 - (R_*/R)^{1/2}$ and R_* is the radius of the central star. [6]

Show that the expression $R^{3/2}/f^4$ (i.e. the left-hand side of the above equation) has a minimum for $R = 16R_*/9$. Use this to determine the minimum mass-accretion rate $\dot{M}_{\rm crit}$ below which there is no hydrostatic solution, expressing your solution in terms of the critical Eddington accretion rate $\dot{M}_{\rm edd}$. [6]

Estimate the ratio of $\dot{M}_{\rm crit}/\dot{M}_{\rm Edd}$ for a cataclysmic variable, containing a white dwarf of $1\,M_\odot$ and radius of 6000 km. Speculate what happens when $\dot{M}$ falls below $\dot{M}_{\rm crit}$. [4]

[The problem is based on the analysis in M. Czerny & A. R. King (1989, MNRAS, 236, 843).]

Section B: Essay Questions

Write a short essay (of the order of 1000 words) on **one** *of the two topics below.*

Essay A: Self-Gravitating Discs

Write an essay on the physics of self-gravitating discs and important applications of these. Specifically, (a) sketch the derivation of the Toomre criterion and explain the role of thermal pressure and rotation in stabilizing a disc, discuss (b) the different situations under which a disc can become Toomre unstable and discuss (c) two potential applications (e.g. the formation of planets/brown dwarfs and gamma-ray bursts). [30]

Essay B: The Magnetorotational Instability

Write an essay on the magnetorotational instability (Balbus-Hawley instability). Specifically, discuss (a) the physical motivation (the viscosity problem), (b) how the instability works, including some simple estimates of its strength (growth rate), (c) applications to accretion discs, and (d) possible limitations of the theory. [30]