Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

DISC ACCRETION IN ASTROPHYSICS: THEORY AND APPLICATIONS

Trinity Term 2019

TUESDAY, 25th JUNE 2019, 12noon to THURSDAY 27th JUNE 2019, 12noon

Answer **all** *of the questions in Section A and* **one** *of the two essay questions from Section B. Answer booklets are provided for you to use but you may type your answers if you wish. Typed answers should be printed single-sided and the pages securely fastened together.*

You may refer to books and other sources when completing the exam but should not discuss the exam with anyone else.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

Section A: Accretion Disc Physics

Please answer **all questions** *in this section.*

1. The continuity equation and the Navier-Stokes equation (in its incompressible form) can be written as

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0,$$

and

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} = -\frac{1}{\rho}\nabla P + \frac{\mathbf{F}}{\rho} + \nu \nabla^2 \mathbf{v}.$$

Explain the origin of these equations and the physical meaning of the individual terms. [6]

Show that, in cylindrical coordinates (R, ϕ, z), for an axi-symmetric system (such as an accretion disc) with only a central external force, the continuity equation and the azimuthal component of the velocity field, v_ϕ, can be written as

$$\frac{\partial \rho}{\partial t} + \frac{1}{R}\frac{\partial (R\rho v_R)}{\partial R} + \frac{\partial (\rho v_z)}{\partial z} = 0,$$

and

$$\frac{\partial v_\phi}{\partial t} = -v_R \frac{\partial v_\phi}{\partial R} - v_z \frac{\partial v_\phi}{\partial z} - \frac{v_\phi v_R}{R} + \nu \left[\frac{1}{R}\frac{\partial}{\partial R}\left(R \frac{\partial v_\phi}{\partial R} \right) + \frac{\partial^2 v_\phi}{\partial z^2} - \frac{v_\phi}{R^2} \right].$$

[You may find find the webpage
"en.wikipedia.org/wiki/Del_in_cylindrical_and_spherical_coordinates" useful.] [4]

Assuming that the dynamical viscosity $\rho\nu$ is spatially uniform, show that these equations can be combined to give

$$\frac{\partial}{\partial t}(\rho R v_\phi) + \nabla \cdot (R\rho v_\phi \mathbf{v} - \rho\nu R^2 \nabla \Omega) = 0,$$

where Ω is the angular velocity. [8]

Interpret this equation physically. [2]

2. Consider a stationary accretion disc around a Schwarzschild black hole of mass M. Explain the importance of the *last stable circular orbit*, $r_{\rm lso}$, the *Eddington limit* and the associated *Eddington accretion rate*. Show that the latter is given by

$$\dot{M}_{\rm edd} = \frac{4\pi c R}{\kappa},$$

where R is the radius (in the case of spherical accretion) and κ the opacity of the gas. [6]

Adopting the Shakura-Sunyaev model for a stationary, optically thick accretion disc (i.e. starting with equations (20)–(24) in the lecture notes), show that, for a disc in which radiation pressure dominates and the opacity is given by pure electron scattering (i.e. $\kappa = 0.034\,\mathrm{m^2/kg}$ for a solar-type plasma), the disc scale height, H, and the radial drift velocity, u, as a function of radius r and accretion rate $\dot{M}$ are given by

$$H = \frac{3\kappa}{8\pi c}\,\dot{M}\,f^4,$$

and

$$\frac{u}{v_{\rm K}} = \frac{1}{r^2}\left(\frac{3\kappa\dot{M}}{8\pi c}\right)^2 f^4\,\alpha,$$

where $v_{\rm K}$ is the local Keplerian velocity and f and α are as defined in the lectures. [10]

Express H and $u/v_{\rm K}$ in terms of $\dot{M}/\dot{M}_{\rm Edd}$, taking R to be $r_{\rm lso}$ in the latter. What happens in the limit $\dot{M} \to \dot{M}_{\rm Edd}$? Comment on the validity of the model in this limit. [4]

3. As was shown in the lectures, in the case of a stationary, optically thick accretion disc radiating as a black body, the effective temperature, $T_{\rm eff}$, as a function of radius, r, is given by

$$\sigma T_{\rm eff}^4 = \frac{3GM\dot{M}}{8\pi r^3}\,f^4, \qquad \text{with} \quad f^4 = 1 - \left(\frac{r_0}{r}\right)^{1/2},$$

where M is the central mass, $\dot{M}$ the mass accretion rate and r_0 the inner radius of the disc.

Consider the mass flow in the outer disc, i.e. at a radius $r \gg r_0$ (so that you can set $f \simeq 1$). Show that the energy released between two radii r_1 and r_2 is three times the energy expected to be released in that region from the virial theorem. Explain this suprising result. [5]

Now consider the disc near the inner radius r_0. Explain why the main contribution to the integral luminosity of the disc comes from the region around r where $\sigma T_{\rm eff}^4 r^2$ is a maximum. Show that this maximum occurs at $r_{\rm max} = 2.25\,r_0$. Explain how this affects the determination of central black-hole masses based on the spectrum of an accretion disc. [5]

4. Consider a local patch in a geometrically thin Keplerian accretion disc with angular velocity Ω in a Newtonian gravitational potential. Assume that the patch is in vertical hydrostatic equilibrium and has an isothermal temperature profile, i.e. $P = c_{\rm s}^2\,\rho$, where p is the pressure, ρ the mass density and $c_{\rm s}$ the sound speed.

Show that the vertical density profile of the disc is given by

$$\rho = \rho_0 e^{-z^2/2h^2},$$

where the scale height h is given by $c_{\rm s}/\Omega$. [6]

Consider a disc where radiation pressure can be important, i.e. the equation of state is given by

$$P = \frac{k\rho T}{\mu m_{\rm H}} + \frac{1}{3}aT^4,$$

where μ is the mean molecular weight and all other variables/constants have their usual meanings. Assuming local thermodynamic equilibrium and an isothermal density profile, show that, for a given surface density Σ, $c_{\rm s}$ is related to the mid-plane temperature $T_{\rm c}$ by

$$c_{\rm s} = \alpha T_{\rm c}^4 + \sqrt{\alpha^2 T_{\rm c}^8 + \beta T_{\rm c}},$$

where α and β should be determined in terms of Σ and physical constants. [8]

Argue why, for a fixed surface density, the vertically integrated heating rate $\dot{Q}_+ \propto c_{\rm s}^2$, while the vertically integrated cooling rate $\dot{Q}_- \propto T_{\rm c}^4$. Show that there is a critical temperature $T_{\rm crit}$ such that the disc is thermally unstable (at constant Σ) when $T_{\rm c} > T_{\rm crit}$, and stable otherwise.

What is the physical relevance of the temperature $T_{\rm crit}$? [6]

Section B: Essay Questions

Write a short essay (of the order of 1000 words) on **one** *of the two topics below.*

Essay A: The Magnetorotational Instability

Write an essay on the magnetorotational instability (Balbus-Hawley instability). Specifically, discuss (a) the physical motivation (the viscosity problem), (b) how the instability works, including some simple estimates of its strength (growth rate), (c) applications to accretion discs, and (d) possible limitations of the theory. [30]

Essay B: General Relativistic Accretion Discs

Write an essay on some of the main effects in accretion physics introduced by General Relativity (as compared to purely Newtonian gravity). Consider specifically the differences in the orbits of test particles and their stability and explain the concept of *frame dragging* and its consequences. In general, consider accretion discs around neutron stars, black holes without spin, black holes with aligned spins and black holes with mis-aligned spins. Discuss how energy can be extracted from a spinning black hole and how this could power relativistic jets. [30]