A15278S1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

COLLISIONLESS PLASMA PHYSICS TAKE HOME EXAM

Hilary Term 2021

Tuesday, 16th March 12 noon 2021, to Thursday, 18th March 12 noon 2021

You should submit answers to all questions.

You may either type of hand-write your answers. You may refer to books and other sources when completing the exam but should not discuss the exam with anyone else.

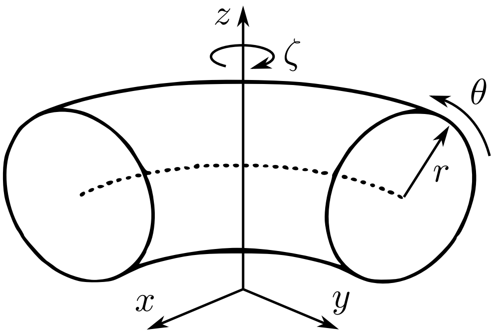

Figure 1: A diagram showing the axisymmetric toroidal coordinates (r, θ, ζ). Only a part of the torus is shown.

1. In this question you will use kinetic MHD to investigate force balance in an axisymmetric toroidal magnetic field $\mathbf{B}$. To describe the toroidal magnetic geometry, we use the toroidal coordinates (r, θ, ζ) shown in figure 1. The coordinates (r, θ, ζ) form a right-handed orthonormal coordinate system, and are defined by their relationship to the Cartesian coordinates (x, y, z):

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} R(r,\theta)\cos\zeta \\ -R(r,\theta)\sin\zeta \\ Z(r,\theta) \end{pmatrix}, \tag{1}$$

where $Z(r, \theta) = r \sin\theta$, and $R(r, \theta) = R_0 + r \cos\theta$, with R_0 the major radius of the torus. The minor radial coordinate r varies between the limits $0 < r \leqslant a$. The unit direction vectors of the (r, θ, ζ) coordinates are

$$\hat{\mathbf{r}} = \begin{pmatrix} \cos\theta\cos\zeta \\ -\cos\theta\sin\zeta \\ \sin\theta \end{pmatrix}, \quad \hat{\boldsymbol{\theta}} = \begin{pmatrix} -\sin\theta\cos\zeta \\ \sin\theta\sin\zeta \\ \cos\theta \end{pmatrix}, \text{ and } \quad \hat{\boldsymbol{\zeta}} = \begin{pmatrix} -\sin\zeta \\ -\cos\zeta \\ 0 \end{pmatrix}. \tag{2}$$

Consider a simple quasineutral plasma consisting of electrons with mass m_e and charge $-e$, and ions with mass m_i and charge $Z_i e$. Assume that the system is in steady state, with electric field $\mathbf{E} = 0$; assume that the distribution function of ions $\langle f_i \rangle_\varphi$ and electrons $\langle f_e \rangle_\varphi$ are *isotropic* in velocity space about their respective mean velocities, $\mathbf{u}_i$ and $\mathbf{u}_e$. Assume that there is negligible mass flow, i.e., $\mathbf{u}_i = 0$.

(a) [5 marks] Use the parallel and perpendicular momentum equations of kinetic MHD to show that the total isotropic pressure P, and the current $\mathbf{J}$ satisfy

$$\mathbf{B} \cdot \nabla P = 0, \tag{3}$$

and

$$\mathbf{J}_\perp \cdot \nabla P = 0. \tag{4}$$

Here, the total isotropic pressure is defined by $P = \sum_s (p_{s\parallel} + 2p_{s\perp})/3$, with $s = (i, e)$ an index that runs over species, and $p_{s\parallel}$ and $p_{s\perp}$ the parallel and perpendicular pressures of the species s, respectively. The perpendicular current $\mathbf{J}_\perp = \mathbf{J} - \mathbf{J}_\parallel$, with $\mathbf{J}_\parallel = (\mathbf{J} \cdot \mathbf{b})\mathbf{b}$, and $\mathbf{b}$ the unit vector in the direction of $\mathbf{B}$. *[Hint: Order $|\mathbf{J}_\parallel| \sim |\mathbf{J}_\perp| \sim P/aB$, with $B = |\mathbf{B}|$, and $2\mu_0 P/B^2 \sim 1$, and determine the allowed size of the electron mean velocity $\mathbf{u}_e$.]*

(b) [10 marks] Axisymmetry implies that the pressure and the components of the magnetic field are independent of ζ, i.e., $P = P(r,\theta)$ and $\mathbf{B} = B_r(r,\theta)\hat{\mathbf{r}} + B_\theta(r,\theta)\hat{\boldsymbol{\theta}} + B_\zeta(r,\theta)\hat{\boldsymbol{\zeta}}$, respectively. Show that an axisymmetric magnetic field may be written in the form

$$\mathbf{B} = \frac{I(\psi)}{R}\hat{\boldsymbol{\zeta}} + \frac{1}{R}\hat{\boldsymbol{\zeta}} \times \nabla\psi, \tag{5}$$

where $R = R(r,\theta)$ and $\psi = \psi(r,\theta)$. To do this, first use $\nabla \cdot \mathbf{B} = 0$ to relate B_r and B_θ to a function $\psi(r,\theta)$. Second, use equation (3) to show that $P = P(\psi)$. Third, use equation (4), and Ampère's law to derive a relation for B_ζ. Finally, compute the integral

$$\frac{1}{2\pi}\int_0^{2\pi}\int_0^r B_\theta(s,\theta)\, R(s,\theta) ds d\zeta,$$

and hence give a physical interpretation for ψ.

Hint: You may use without proof the forms of grad, div, and curl in the toroidal (r,θ,ζ) coordinates. For a scalar ϕ and vector $\mathbf{U} = U_r\hat{\mathbf{r}} + U_\theta\hat{\boldsymbol{\theta}} + U_\zeta\hat{\boldsymbol{\zeta}}$, these are

$$\nabla\phi = \hat{\mathbf{r}}\frac{\partial\phi}{\partial r} + \frac{\hat{\boldsymbol{\theta}}}{r}\frac{\partial\phi}{\partial\theta} + \frac{\hat{\boldsymbol{\zeta}}}{R}\frac{\partial\phi}{\partial\zeta},$$

$$\nabla \cdot \mathbf{U} = \frac{1}{rR}\frac{\partial}{\partial r}(rRU_r) + \frac{1}{rR}\frac{\partial}{\partial\theta}(RU_\theta) + \frac{1}{R}\frac{\partial U_\zeta}{\partial\zeta},$$

and

$$\nabla \times \mathbf{U} = \frac{\hat{\mathbf{r}}}{rR}\left(\frac{\partial(RU_\zeta)}{\partial\theta} - \frac{\partial(rU_\theta)}{\partial\zeta}\right) + \frac{\hat{\boldsymbol{\theta}}}{R}\left(\frac{\partial U_r}{\partial\zeta} - \frac{\partial(RU_\zeta)}{\partial r}\right) + \frac{\hat{\boldsymbol{\zeta}}}{r}\left(\frac{\partial(rU_\theta)}{\partial r} - \frac{\partial U_r}{\partial\theta}\right).$$

(c) [15 marks] Consider the kinetic MHD perpendicular momentum equation. In the limit of

$$\frac{B_\theta}{B_\zeta} \sim \epsilon = \frac{a}{R_0} \ll 1 \sim \frac{2\mu_0 p}{B_\theta^2}, \text{ with } \frac{dI^2}{dr} \sim \mu_0 R_0^2\frac{p}{a},$$

and $\psi = \psi_0(r) + \psi_1(r,\theta) + O(\epsilon^2)$, show that leading-order force balance becomes

$$\mu_0 R_0^2\frac{dP(\psi_0)}{dr} + I(\psi_0)\frac{dI(\psi_0)}{dr} + \frac{1}{r}\frac{d\psi_0}{dr}\frac{d}{dr}\left(r\frac{d\psi_0}{dr}\right) = 0. \tag{6}$$

Briefly interpret this result.

(d) [5 marks] Find ψ_0 by solving equation (6) for I a constant, $dP/d\psi_0 = -2I/q\mu_0 R_0^3$, where q is a constant, and subject to no divergences at $r = 0$. Give $\mathbf{B}$ to $O(\epsilon)$.

2. In this question you will consider particle motion in axisymmetric toroidal magnetic geometry, shown in figure 1. The (r, θ, ζ) toroidal coordinates are defined in equation (1), and the unit vectors $\hat{\mathbf{r}}$, $\hat{\boldsymbol{\theta}}$, and $\hat{\boldsymbol{\zeta}}$ are defined in equation (2). Assume that the plasma is in steady state, the electric field $\mathbf{E} = 0$, and take the magnetic field to be

$$\mathbf{B} = \frac{I}{R_0}\left(1 - \frac{r}{R_0}\cos\theta\right)\hat{\boldsymbol{\zeta}} + \frac{\hat{\boldsymbol{\zeta}}}{R_0} \times \nabla\Psi, \tag{7}$$

where I is a constant (with units current/μ_0), R_0 is the major radius, $r \sim a$, with a the minor radius, and

$$\Psi(r) = \frac{B_0 R_0 r^4}{4a^3},$$

with $B_0 \sim aI/R_0^2$ a constant magnetic field. Assume throughout that $\epsilon = a/R_0 \ll 1$.

(a) [3 marks] Consider particle motion in a magnetic field of the form (7). Use the guiding centre equations to show that Ψ is conserved by particle motion on time scales of $O(a/v_{th})$, where v_{th} is the thermal speed.

(b) [2 marks] Show that the kinetic energy of particles is conserved.

(c) [10 marks] Keeping terms to $O(\epsilon)$, sketch the magnetic field strength $B(\theta) = |\mathbf{B}|$ (at fixed $r \neq 0$). Hence, describe the trajectories of particles on time scales of $O(a/v_{th})$.

(d) [15 marks] Write down the drift kinetic equation valid for $O(a/v_{th})$ time scales. A trace impurity ion species is introduced into the device, and heated by cyclotron waves. The resulting distribution of the impurity species at $\theta = 0$ is

$$\langle f \rangle_\varphi = N\left(\frac{m}{2\pi T}\right)^{3/2}\left(\frac{m\mu B(r, \theta = 0)}{T}\right)^2 \exp\left(-\frac{mv_\parallel^2/2 + m\mu B(r, \theta = 0)}{T}\right),$$

where N and T are constant densities and temperatures, respectively, m is the particle mass, $v_\parallel$ is the parallel velocity and μ is the magnetic moment. Calculate the density of impurity ions at all (r, θ), correct to $O(\epsilon)$.

3. In this question you will consider the propagation of cold plasma waves in a sheared-slab magnetic geometry. Assume that the magnetic field has the form

$$\mathbf{B} = \frac{B_0}{(1+(x/L)^2)^{1/2}} \left(\hat{\mathbf{z}} + \frac{x}{L}\hat{\mathbf{y}}\right),$$

where (x, y, z) are the usual Cartesian coordinates, with corresponding unit vectors $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$, B_0 is a constant magnetic field, and L is a constant length. The plasma consists of ions, with mass m_i and charge $Z_i e$, and electrons, with mass m_e and charge $-e$. Assume throughout that $Z_i \sim 1 \ll m_i/m_e$. The electron density n_e is constant throughout the plasma, and the plasma equilibrium is quasineutral. An extraordinary mode (X mode) with frequency $\omega > \sqrt{\omega_{pe}^2 + \Omega_e^2}$ and wave vector $\mathbf{k} = k_{x0}\hat{\mathbf{x}} + k_{y0}\hat{\mathbf{y}}$ is launched from $x = 0$. Here, the plasma frequency ω_{pe} and the cyclotron frequency Ω_e satisfy

$$\omega_{pe} = \sqrt{\frac{e^2 n_e}{\epsilon_0 m_e}} \gtrsim \Omega_e = \frac{eB_0}{m_e}.$$

(a) [5 marks] By considering the ray-tracing equation, show that $k_y = \mathbf{k}\cdot\hat{\mathbf{y}}$ and $k_z = \mathbf{k}\cdot\hat{\mathbf{z}}$ are constants along the path of the wave.

(b) [3 marks] Determine $k_\parallel = \mathbf{k}\cdot\mathbf{b}$ and $\mathbf{k}_\perp = \mathbf{k} - k_\parallel\mathbf{b}$, with $\mathbf{b}$ the unit vector in the direction of the magnetic field. Hence, write down the basis vectors $\mathbf{k}_\perp/|\mathbf{k}_\perp|$ and $\mathbf{b}\times\mathbf{k}_\perp/|\mathbf{k}_\perp|$.

(c) [12 marks] Show that the cold plasma dispersion relation for general x can be written in the form

$$\left(\frac{ck_x}{\omega}\right)^4 G(x, k_y, \omega) + \left(\frac{ck_x}{\omega}\right)^2 H(x, k_y, \omega) + K(x, k_y, \omega) = 0. \quad (8)$$

Determine G, H, and K. For the propagating X mode, use equation (8), and the initial condition at $x = 0$, to find the function $F(x, k_y, \omega)$ such that

$$\left(\frac{ck_x}{\omega}\right)^2 = F(x, k_y, \omega).$$

(d) [3 marks] Write down the condition for the wave to resonate. Show that there is no x at which the X mode resonates.

(e) [6 marks] Write down the condition for the X mode to reflect. Show that if the wave reflects, it reflects at the position

$$\frac{x_c}{L} = \left(\frac{(n_y^2/\epsilon_\parallel - 1)(\epsilon_\perp(\epsilon_\perp - n_y^2) - g^2)}{(n_y^2 - \epsilon_\perp)^2 - g^2}\right)^{1/2},$$

where $n_y = ck_{y0}/\omega$, and $\epsilon_\parallel$, $\epsilon_\perp$ and g are components of the cold plasma dielectric tensor. Use your result to show that the wave cannot reflect if $k_{y0} = 0$, or if $\Omega_e \ll \omega$.

(f) [6 marks] Finally, assume that $\omega = 2\omega_{pe}$, and $\omega_{pe} = \sqrt{2}\Omega_e$, and determine the range of k_{y0} for which reflection occurs.