Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

COLLISIONLESS PLASMA PHYSICS

Trinity Term 2022

Tuesday, 21st JUNE 2022, 12:00

You should submit answers to all three questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

In the following two questions, we consider a plasma consisting of electrons with mass m_e, charge $-e$, number density n_e and temperature T_e, and of ions with mass m_i, charge Ze, number density n_i and temperature T_i. The electrons' and ions' thermal speeds are v_{the} and $v_{\text{th}i}$, respectively. A uniform magnetic field $\boldsymbol{B} = B\hat{\boldsymbol{z}}$ is imposed, where $\hat{\boldsymbol{z}}$ is a unit vector in the standard Cartesian coordinate system (x, y, z). You may assume that

$$\frac{m_e}{m_i} \ll 1 \sim Z \sim \frac{T_e}{T_i} \sim \frac{\omega_{\text{pe}}}{\Omega_e}, \tag{1}$$

where ω_{pe} is the electron plasma frequency and Ω_e is the electron cyclotron frequency.

1. Consider an inhomogeneous plasma in a uniform magnetic field, as described above. The equilibrium electron density is $n_e(x)$ with $\mathrm{d}n_e/\mathrm{d}x > 0$ within the plasma. An antenna at $x = 0$ launches a wave with wavenumber $\boldsymbol{k} = k_x\hat{\boldsymbol{x}} + k_y\hat{\boldsymbol{y}}$, where $k_x, k_y > 0$, and frequency $\omega \gtrsim \omega_{\text{pe}} \gg kv_{\text{the}}$.

   (a) [2 marks] Write down the cold-plasma dielectric tensor $\boldsymbol{\epsilon}$ in terms of its components $\epsilon_\perp$, g, and $\epsilon_\parallel$.

   (b) [4 marks] Starting from the cold-plasma dispersion relation

   $$\left[n^2(\hat{\boldsymbol{k}}\hat{\boldsymbol{k}} - \boldsymbol{I}) + \boldsymbol{\epsilon}\right] \cdot \delta\boldsymbol{E} = 0,$$

   show that there are two solutions for the index of refraction $n = kc/\omega$ (where c is the speed of light) for a wave with $\boldsymbol{k} \cdot \boldsymbol{B} = 0$: the ordinary (O-mode) with $n^2 = \epsilon_\parallel$ and the extraordinary (X-mode) with $n^2 = \epsilon_\perp - g^2/\epsilon_\perp$.

   (c) [2 marks] What distinguishes the ordinary and extraordinary modes? What is 'ordinary' about the former?

   (d) [2 marks] Using ray tracing, show that k_y is constant along the path of the wave.

   (e) [4 marks] Consider the O-mode in the plasma described above. Assuming the ordering (1), show that the wave will be reflected at the position x that satisfies

   $$n_e(x) = \epsilon_0 m_e(\omega^2 - k_y^2 c^2)/e^2.$$

   Sketch the trajectory of the reflected ray.

   (f) [10 marks] Now consider the X-mode. Again using the ordering (1), show that the mode satisfies

   $$k^2c^2 = \frac{[\omega^2 - \omega_{\text{pe}}^2(x)]^2 - \Omega_e^2\omega^2}{\omega^2 - \omega_{\text{pe}}^2(x) - \Omega_e^2}.$$

   What does the limit $\Omega_e \to 0$ correspond to? How is it related to the O-mode?

   (g) [6 marks] Show that for a weakly magnetised plasma, i.e., one with $\Omega_e \ll \omega \sim \omega_{pe}$, an X-mode with wavenumber k_y and frequency ω will follow the same path as an O-mode with the same frequency but with

   $$k_{y,\text{O}}^2 = k_y^2 + \frac{\omega_{\text{pe}}^2\Omega_e^2}{(\omega^2 - \omega_{\text{pe}}^2)c^2}.$$

   Deduce that the X-mode is reflected earlier, i.e., at a lower density n_e, than the O-mode. Why is this unsurprising?

2. Consider a homogeneous plasma in a Maxwellian equilibrium, viz., for the species α (= e electrons and = i ions),

$$f_\alpha = \frac{n_\alpha}{\pi^{3/2} v_{\mathrm{th}\alpha}^3} e^{-v^2/v_{\mathrm{th}\alpha}^2}.$$

(a) [6 marks] Show that, in Cartesian (x, y, z) coordinates, the dielectric tensor $\boldsymbol{\epsilon}$ for a wave with wavenumber $\boldsymbol{k} = k_\parallel \hat{\boldsymbol{z}}$ and frequency ω is given by

$$\boldsymbol{\epsilon} = \begin{pmatrix} \epsilon_\perp & ig & 0 \\ -ig & \epsilon_\perp & 0 \\ 0 & 0 & \epsilon_\parallel \end{pmatrix},$$

where

$$\epsilon_\perp = 1 + \frac{1}{2} \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega |k| v_{\mathrm{th}\alpha}} \left[Z\left(\frac{\omega - \Omega_\alpha}{|k_\parallel| v_{\mathrm{th}\alpha}}\right) + Z\left(\frac{\omega + \Omega_\alpha}{|k_\parallel| v_{\mathrm{th}\alpha}}\right) \right],$$

$$\epsilon_\parallel = 1 + \sum_\alpha \frac{2\omega_{\mathrm{p}\alpha}^2}{k_\parallel^2 v_{\mathrm{th}\alpha}^2} \left[1 + \frac{\omega}{|k_\parallel| v_{\mathrm{th}\alpha}} Z\left(\frac{\omega}{|k_\parallel| v_{\mathrm{th}\alpha}}\right) \right],$$

$$g = \frac{1}{2} \sum_\alpha \frac{\omega_{\mathrm{p}\alpha}^2}{\omega |k_\parallel| v_{\mathrm{th}\alpha}} \left[Z\left(\frac{\omega - \Omega_\alpha}{|k_\parallel| v_{\mathrm{th}\alpha}}\right) - Z\left(\frac{\omega + \Omega_\alpha}{|k_\parallel| v_{\mathrm{th}\alpha}}\right) \right].$$

Here $\omega_{\mathrm{p}\alpha}$ and Ω_α are the plasma frequency and the cyclotron frequency, respectively, associated with species α and

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_{C_L} \mathrm{d}u \, \frac{e^{-u^2}}{u - \zeta}$$

is the plasma dispersion function.

(b) [4 marks] Hence show that the dispersion relation for a wave propagating along $\boldsymbol{B}$ with perpendicular polarisation ($\delta \boldsymbol{E} \cdot \boldsymbol{B} = 0$) is

$$\frac{k_\parallel^2 c^2}{\omega^2} = 1 + \sum_s \frac{\omega_{\mathrm{p}\alpha}^2}{\omega |k_\parallel| v_{\mathrm{th}\alpha}} Z\left(\frac{\omega \mp \Omega_\alpha}{|k_\parallel| v_{\mathrm{th}\alpha}}\right). \tag{2}$$

(c) [6 marks] Define the cold-plasma limit. Show that in this limit, the dispersion relation (2) reduces to

$$D(\omega) \equiv -\frac{k_\parallel^2 c^2}{\omega^2} + \frac{(\omega \mp \omega_{\mathrm{L}})(\omega \pm \omega_{\mathrm{R}})}{(\omega \pm \Omega_e)(\omega \mp \Omega_i)} + \sum_\alpha \frac{i\sqrt{\pi}\omega_{\mathrm{p}\alpha}^2}{\omega |k_\parallel| v_{\mathrm{th}\alpha}} \exp\left[-\frac{(\omega \mp \Omega_\alpha)^2}{k_\parallel^2 v_{\mathrm{th}\alpha}^2} \right] = 0, \tag{3}$$

where the electron gyrofrequency Ω_e is defined to be positive. Give expressions for the frequencies ω_{L} and ω_{R}.

(d) [6 marks] Consider the low-frequency ($\omega \ll \Omega_e, \Omega_i$) cold-plasma solutions to (3). Let $\omega = \omega_0 + i\gamma$, where $\omega_0, \gamma \in \mathbb{R}$. Argue that $\omega_0 \gg \gamma$ and that $\omega_0 \approx \pm k_\parallel v_{\mathrm{A}}$, where v_{A} is the Alfvén speed. Provide an expression for v_{A} in terms of $\omega_{\mathrm{p}i}$, Ω_i, and c.

(e) [8 marks] Argue that the damping rate is given by

$$\gamma \approx -\mathrm{Im} D(\omega_0) \left[\frac{\partial}{\partial \omega_0} \mathrm{Re} D(\omega_0) \right]^{-1}.$$

Hence show that

$$\gamma \approx -\frac{\sqrt{\pi}\Omega_i^2}{2|k_\parallel| v_{\mathrm{th}i}} e^{-\Omega_i^2/k_\parallel^2 v_{\mathrm{th}i}^2}.$$

3. The total energy of a plasma is

$$\mathscr{E} = \int \mathrm{d}^3\boldsymbol{r} \left[\sum_\alpha \left(\frac{m_\alpha n_\alpha u_\alpha^2}{2} + \frac{3}{2} p_\alpha \right) + \frac{B^2}{8\pi} \right], \tag{4}$$

where $\boldsymbol{B}$ is the magnetic field, $B = |\boldsymbol{B}|$, and, for each species α, m_α is the particle mass, n_α the number density, $\boldsymbol{u}_\alpha$ the mean velocity, $p_\alpha = (1/3)\int \mathrm{d}^3\boldsymbol{w}\, m_\alpha w^2 f_\alpha$ the total pressure, and f_α the distribution function defined with respect to the peculiar velocity $\boldsymbol{w} = \boldsymbol{v} - \boldsymbol{u}_\alpha$.

(a) [20 marks] Assume that the plasma obeys the Kinetic Magnetohydrodynamics (KMHD) approximation. Starting from the KMHD equations, prove that the total energy is conserved, viz.,

$$\frac{\mathrm{d}\mathscr{E}}{\mathrm{d}t} = 0, \tag{5}$$

assuming no energy flows through any boundaries of the system. You may use any set of kinetic variables that you consider convenient for this purpose. If you find handling the general case with species drifts challenging, you may assume that there are no species drifts. You may also use without derivation any result already proven in the Lecture Notes.

(b) [10 marks] Define the temperature of species α to be $T_\alpha = p_\alpha/n_\alpha$. Show that its evolution equation can be written in the following form:

$$\frac{3}{2} n_\alpha \frac{\mathrm{d}T_\alpha}{\mathrm{d}t_\alpha} = p_{\parallel\alpha} \frac{1}{n_\alpha}\frac{\mathrm{d}n_\alpha}{\mathrm{d}t_\alpha} - \boldsymbol{\nabla}\cdot\left(q_\alpha \hat{\boldsymbol{b}}\right) + \left(p_{\perp\alpha} - p_{\parallel\alpha}\right)\frac{1}{B}\frac{\mathrm{d}B}{\mathrm{d}t_\alpha}, \tag{6}$$

where $\hat{\boldsymbol{b}} = \boldsymbol{B}/B$, $\mathrm{d}/\mathrm{d}t_\alpha = \partial/\partial t + \boldsymbol{u}_\alpha \cdot \boldsymbol{\nabla}$ is the convective derivative with respect to the mean motion of species α, and you may ignore any heating or cooling arising from collisions with particles of other species. Explain what q_α, $p_{\perp\alpha}$ and $p_{\parallel\alpha}$ are. Interpret the first three terms on the left-hand side of the above equation.

(c) [10 marks] Let the rate of relaxation of pressure anisotropy of species α due to particle collisions be ν_α. Assuming this rate to be larger than the rate of dynamical evolution of all quantities and assuming also an incompressible mass flow, no species drifts, and no heat fluxes, show that the local rate of viscous heating of the plasma is nowhere negative and is non-zero in all fluid elements where the magnetic field's strength is changing.