A15278S1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

COLLISIONLESS PLASMA PHYSICS
TAKE-HOME EXAM

HILARY TERM 2018

TUESDAY, 13TH MARCH 2018, 12noon to THURSDAY, 15 MARCH 2018, 12noon

You should submit answers to all questions. Answer booklets are provided for you to use but you may type your answers if you wish. Typed answers should be printed single-sided and the pages securely fastened together.

You may refer to books and other sources when completing the exam but should not discuss the exam with anyone else.

Do not turn this page until you are told that you may do so

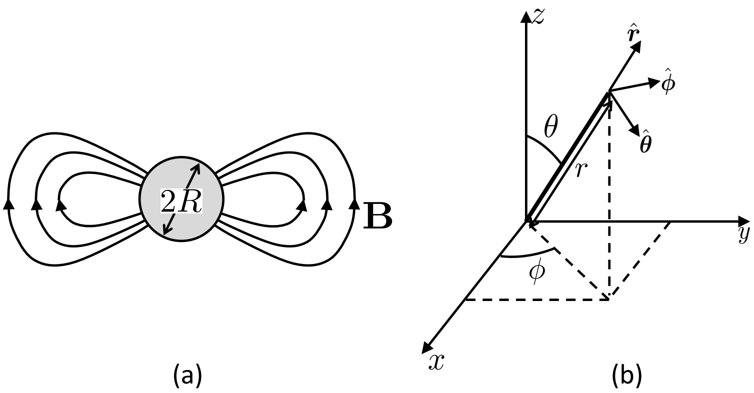

Figure 1: (a) Dipolar magnetic field around a planet. (b) Spherical coordinates.

1. Consider the dipolar magnetic field around a spherical planet of radius R shown in figure 1(a). The magnetic field of a dipole is

$$\mathbf{B}(r,\theta) = B_r\hat{\mathbf{r}} + B_\theta\hat{\boldsymbol{\theta}} = -\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\hat{\mathbf{r}} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\hat{\boldsymbol{\theta}}, \tag{1}$$

where the flux function $\psi(r,\theta)$ is

$$\psi(r,\theta) = \frac{A\sin^2\theta}{r}. \tag{2}$$

Here $\{r,\theta,\phi\}$ are the usual spherical coordinates, shown in figure 1(b), and $\{\hat{\mathbf{r}},\hat{\boldsymbol{\theta}},\hat{\boldsymbol{\phi}}\}$ are the usual unit vectors associated to the spherical coordinates, also shown in figure 1(b).

Particles of charge Ze and mass m move in the dipolar magnetic field (1). The electric field $\mathbf{E}$ is negligibly small. Assume that the particles are magnetized, that is, their characteristic gyroradius ρ is small compared to the characteristic size of the system, $\rho/R \ll 1$. Using the lowest order guiding centre equations, answer the following questions.

(a) [5 marks] Particles move along the magnetic field line on which they started. Show that this is equivalent to moving along lines of constant $\psi(r,\theta)$ and constant ϕ. Show that the kinetic energy $K = mv^2/2$ of each particle is conserved.

(b) [10 marks] To determine the position of each particle, we use the coordinates $\{\psi,\theta,\phi\}$. Plot B as a function of θ for fixed ψ and ϕ, and using this plot, describe the motion of a particle with kinetic energy K and magnetic moment μ. Give a formula for the points θ_b in which the parallel velocity vanishes.

(c) [5 marks] Show that particles with $\psi \sim |A|/L$ and $L \gg R$ collide with the planet when

$$\frac{m\mu}{K} \leqslant \frac{R^3}{2|A|}. \tag{3}$$

(d) [10 marks] Assume that any particle that collides with the planet is lost, and hence there are no particles that satisfy condition (3). Using the lowest order drift kinetic equation, calculate the distribution function of the particles $f(r, \theta, \mathbf{v})$ everywhere given the distribution function at $\theta = \pi/2$, which is

$$f(r, \theta = \pi/2, \mathbf{v}) = N \left(\frac{m}{2\pi T} \right)^{3/2} \exp \left(-\frac{mv^2}{2T} \right). \tag{4}$$

for $m\mu/K > R^3/2|A|$ and $L < r < 2L$, and

$$f(r, \theta = \pi/2, \mathbf{v}) = 0 \tag{5}$$

otherwise. Here the density N, the temperature T and the length L are constants. Using the distribution function that you have obtained, calculate the density $n = \int f \, \mathrm{d}^3 v$ everywhere, and in particular near $r = R$.

2. By answering the following questions, you are going to determine the behavior of whistler waves in the plasma confined by the Earth's dipolar magnetic field. Consider a plasma composed of one ion species with charge Ze and mass m_i, and electrons with charge $-e$ and mass m_e.

(a) [10 marks] Whistler waves are characterized by frequencies $\sqrt{\Omega_i \Omega_e} \ll \omega \ll \Omega_e$, where $\Omega_i = ZeB/m_i$ and $\Omega_e = eB/m_e$ are the ion and electron gyrofrequencies. Assume $\omega_{pe} \gg \Omega_e$ and $\sqrt{m_e/m_i} \ll 1$, where $\omega_{pe} = \sqrt{e^2 n_e / \epsilon_0 m_e}$ is the electron plasma frequency, n_e is the electron number density and ϵ_0 is the vacuum permittivity. Show that for whistler waves, the cold plasma dielectric tensor becomes

$$\boldsymbol{\epsilon} = \begin{pmatrix} \omega_{pe}^2/\Omega_e^2 & -\mathrm{i}\omega_{pe}^2/\Omega_e \omega & 0 \\ \mathrm{i}\omega_{pe}^2/\Omega_e \omega & \omega_{pe}^2/\Omega_e^2 & 0 \\ 0 & 0 & -\omega_{pe}^2/\omega^2 \end{pmatrix}. \tag{6}$$

When you neglect terms (for example, the ion contributions to the cold plasma dielectric tensor), justify your decision with order of magnitude estimates.

(b) [8 marks] Assuming that $kd_e \ll 1$, where $d_e = c/\omega_{pe}$ is the electron skin depth, show that the whistler wave satisfies the dispersion relation

$$\omega = k_{\|} k d_e^2 \Omega_e. \tag{7}$$

(c) [2 marks] Whistlers get their name from their dispersive nature: waves are generated in the ionosphere and each frequency propagates towards observers in Earth at a different velocity, leading to a signal with time-dependent frequency or "whistle". Argue that the group velocity of the whistlers approximately scales as $\sqrt{\omega}$ and hence observers on Earth receive a signal with a frequency that decreases with time.

(d) [10 marks] Whistler propagates towards Earth along overdense plasma regions that are aligned with the dipolar magnetic field. To study this propagation, consider a system with a uniform magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$ and electron density $n_e(x) = n_{e0}(1 - x^2/L^2)$, where $\{x, y, z\}$ are the usual Cartesian coordinates, and $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ are the unit vectors in the directions of the Cartesian axes. The whistler wave is launched from the origin with wavevector $\mathbf{k} = k_{\perp 0}\hat{\mathbf{x}} + k_{\|0}\hat{\mathbf{z}}$. Using ray tracing equations, show that $k_{\|}$ is constant along rays and find $\mathbf{k}$ as a function of x. Describe the propagation of the whistler wave.

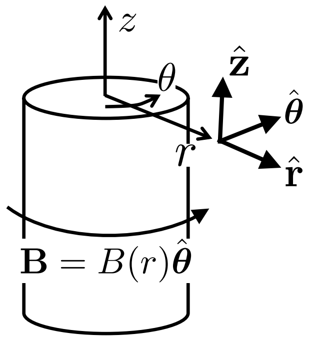

Figure 2: z-pinch and cylindrical coordinates.

3. The z-pinch is a magnetic confinement cylindrical configuration in which the magnetic field closes in azimuthal loops (see Figure 2). The magnitude of the magnetic field depends on radius, that is, $\mathbf{B} = B(r)\hat{\boldsymbol{\theta}}$. Here we use the usual cylindrical coordinates $\{r, \theta, z\}$ and their associated unit vectors $\{\hat{\mathbf{r}}, \hat{\boldsymbol{\theta}}, \hat{\mathbf{z}}\}$, shown in figure 2.

   The plasma contained in the z-pinch is composed of electrons with charge $-e$ and mass m_e, and one ion species with charge Ze and mass m_i. The distribution functions for both species ($s = i, e$) are Maxwellians that only depend on the radial position,

$$\langle f_s \rangle_\varphi = f_{Ms}(r, v_\parallel, \mu) = n_s(r) \left(\frac{m_s}{2\pi T_s(r)} \right)^{3/2} \exp\left(-\frac{m_s(v_\parallel^2/2 + \mu B(r))}{T_s(r)} \right). \tag{8}$$

   The equilibrium electric field is zero, $\mathbf{E} = 0$.

   We consider the stability of this configuration to axisymmetric kinetic MHD modes, that is, to infinitely small perturbations of the form

$$\begin{aligned} \delta\langle f_s \rangle_\varphi &= \tilde{g}_s(r, v_\parallel, \mu) \exp(\mathrm{i}kz - \mathrm{i}\omega t), \\ \delta\mathbf{B} &= \tilde{\mathbf{B}}(r) \exp(\mathrm{i}kz - \mathrm{i}\omega t), \\ \delta\mathbf{v}_E &= \tilde{\mathbf{v}}_E(r) \exp(\mathrm{i}kz - \mathrm{i}\omega t), \\ \delta E_\parallel &= \tilde{E}_\parallel(r) \exp(\mathrm{i}kz - \mathrm{i}\omega t). \end{aligned}$$

   (a) [3 marks] Before considering the perturbations, show that the equilibrium must satisfy the equation

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(P + \frac{B^2}{2\mu_0} \right) + \frac{B^2}{\mu_0 r} = 0, \tag{9}$$

   where $P = n_i T_i + n_e T_e$ is the total plasma pressure.

   (b) [7 marks] Using the perpendicular plasma displacement $\tilde{\boldsymbol{\xi}}_\perp = \tilde{\xi}_r \hat{\mathbf{r}} + \tilde{\xi}_z \hat{\mathbf{z}} = \tilde{\mathbf{v}}_E/(-\mathrm{i}\omega)$, show that the kinetic MHD induction equation gives

$$\tilde{\mathbf{B}} = -\left[\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\tilde{\xi}_r \right) + \mathrm{i}k\tilde{\xi}_z - \left(\frac{2}{r} + \frac{\mu_0}{B^2}\frac{\mathrm{d}P}{\mathrm{d}r} \right) \frac{\tilde{\xi}_r}{r} \right] B\hat{\boldsymbol{\theta}}. \tag{10}$$

   [Hint: use the result derived in part (a) to eliminate $\mathrm{d}B/\mathrm{d}r$ from the equation.] Show that the perturbation δB to the magnitude of the magnetic field is δB_θ, and that the perturbation $\delta\boldsymbol{\kappa}$ to the magnetic field line curvature $\boldsymbol{\kappa} = \hat{\mathbf{b}} \cdot \nabla\hat{\mathbf{b}}$ vanishes.

(c) [10 marks] Using the kinetic MHD drift kinetic equation, obtain the distribution function $\tilde{g}_s$. Integrate it to obtain the perturbed parallel pressures,

$$\begin{aligned}\tilde{p}_{s\|} &= \int m_s v_\|^2 \tilde{g}_s \, \mathrm{d}^3 v + \frac{\tilde{B}_\theta}{B} n_s T_s \\ &= -\left[n_s T_s \left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\tilde{\xi}_r\right) + \mathrm{i}k\tilde{\xi}_z \right) + \left(\frac{2n_s T_s}{r} + \frac{\mathrm{d}(n_s T_s)}{\mathrm{d}r} \right) \tilde{\xi}_r \right],\end{aligned} \tag{11}$$

and the perturbed perpendicular pressures,

$$\begin{aligned}\tilde{p}_{s\perp} &= \int m_s \mu B \tilde{g}_s \, \mathrm{d}^3 v + \frac{2\tilde{B}_\theta}{B} n_s T_s \\ &= -\left[2n_s T_s \left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\tilde{\xi}_r\right) + \mathrm{i}k\tilde{\xi}_z \right) + \left(-\frac{n_s T_s}{r} + \frac{\mathrm{d}(n_s T_s)}{\mathrm{d}r} \right) \tilde{\xi}_r \right].\end{aligned} \tag{12}$$

(d) [5 marks] Show that the kinetic MHD momentum conservation equation gives

$$\begin{aligned}-\omega^2 n_i m_i \tilde{\xi}_r = {} & \frac{\mathrm{d}}{\mathrm{d}r}\left[\left(\frac{B^2}{\mu_0} + 2P \right) \left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\tilde{\xi}_r\right) + \mathrm{i}k\tilde{\xi}_z \right) - \left(\frac{2B^2}{\mu_0} + P \right) \frac{\tilde{\xi}_r}{r} \right] \\ & + \frac{1}{r}\left[\left(\frac{2B^2}{\mu_0} + P \right) \left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\tilde{\xi}_r\right) + \mathrm{i}k\tilde{\xi}_z \right) - \left(\frac{4B^2}{\mu_0 r} + \frac{3P}{r} + 2\frac{\mathrm{d}P}{\mathrm{d}r} \right) \tilde{\xi}_r \right],\end{aligned} \tag{13}$$

$$-\omega^2 n_i m_i \tilde{\xi}_z = \mathrm{i}k \left[\left(\frac{B^2}{\mu_0} + 2P \right) \left(\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\tilde{\xi}_r\right) + \mathrm{i}k\tilde{\xi}_z \right) - \left(\frac{2B^2}{\mu_0} + P \right) \frac{\tilde{\xi}_r}{r} \right]. \tag{14}$$

(e) [10 marks] To solve the equations in part (d), we use a variational principle. Show that the eigenvalues ω^2 are extrema of the functional

$$\Lambda[\tilde{\eta}_r, \tilde{\eta}_z] = \frac{N[\tilde{\eta}_r, \tilde{\eta}_z]}{D[\tilde{\eta}_r, \tilde{\eta}_z]}. \tag{15}$$

where

$$\begin{aligned}N[\tilde{\eta}_r, \tilde{\eta}_z] = {} & \int_0^\infty \left(\frac{B^2}{\mu_0} + 2P \right) \left| \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}(r\tilde{\eta}_r) + \mathrm{i}k\tilde{\eta}_z - \frac{2B^2 + \mu_0 P}{B^2 + 2\mu_0 P}\frac{\tilde{\eta}_r}{r} \right|^2 r\,\mathrm{d}r \\ & + \int_0^\infty \left(\frac{7B^2 + 5\mu_0 P}{B^2 + 2\mu_0 P}\frac{P}{r} + 2\frac{\mathrm{d}P}{\mathrm{d}r} \right) |\tilde{\eta}_r|^2 r\,\mathrm{d}r,\end{aligned} \tag{16}$$

$$D[\tilde{\eta}_r, \tilde{\eta}_z] = \int_0^\infty n_i m_i (|\tilde{\eta}_r|^2 + |\tilde{\eta}_z|^2) r\,\mathrm{d}r. \tag{17}$$

In other words, show that for the solutions $\tilde{\eta}_r = \tilde{\xi}_r$ and $\tilde{\eta}_z = \tilde{\xi}_z$, $\Lambda[\tilde{\xi}_r, \tilde{\xi}_z] = \omega^2$ and that $\Lambda[\tilde{\xi}_r + \delta\tilde{\eta}_r, \tilde{\xi}_z + \delta\tilde{\eta}_z] - \Lambda[\tilde{\xi}_r, \tilde{\xi}_z] = 0$ to first order in $\delta\tilde{\eta}_r \ll \tilde{\xi}_r$ and $\delta\tilde{\eta}_z \ll \tilde{\xi}_z$.

(f) [5 marks] The smallest eigenvalue is the minimum of the functional $\Lambda[\tilde{\eta}_r, \tilde{\eta}_z]$. Hence, one possible strategy to prove that the plasma is unstable, that is, that at least one eigenvalue satisfies $\omega^2 < 0$, is to find functions $\tilde{\eta}_r$ and $\tilde{\eta}_z$ that make $\Lambda[\tilde{\eta}_r, \tilde{\eta}_z]$ negative. Argue that $D[\tilde{\eta}_r, \tilde{\eta}_z]$ is always positive. Then minimize $N[\tilde{\eta}_r, \tilde{\eta}_z]$ by choosing an appropriate function $\tilde{\eta}_z$. The final result should be that $N[\tilde{\eta}_r, \tilde{\eta}_z]$ and hence $\Lambda[\tilde{\eta}_r, \tilde{\eta}_z]$ become negative for sufficiently large $-\mathrm{d}P/\mathrm{d}r$, and as a result, the plasma is unstable for sufficiently large $-\mathrm{d}P/\mathrm{d}r$. Find the critical value of $-\mathrm{d}P/\mathrm{d}r$ for which the plasma becomes unstable.