A15423S1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

COLLISIONAL PLASMA PHYSICS

Trinity Term 2019

TUESDAY, 25th JUNE 2019, 12noon to THURSDAY 27th JUNE 2019, 12noon

*You should submit answers to **all** questions.*
Answer booklets are provided for you to use but you may type your answers if you wish. Typed answers should be printed single-sided and the pages securely fastened together.

You may refer to books and other sources when completing the exam but should not discuss the exam with anyone else.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. Radiofrequency (RF) waves are routinely used to inject momentum and energy into the electrons of a plasma. Consider a steady-state, spatially-homogeneous plasma composed of one ion species with charge Ze and mass m_i and electrons with charge $-e$ and mass m_e. Let v_x, v_y and v_z be the usual Cartesian coordinates for velocity space, and $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$ and $\hat{\mathbf{z}}$ be the unit vectors parallel to the Cartesian axes. Let v, α and β be the spherical coordinates for velocity space depicted in figure 1, and $\hat{\mathbf{v}}$, $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$ be the unit vectors parallel to $\nabla_v v$, $\nabla_v \alpha$ and $\nabla_v \beta$.

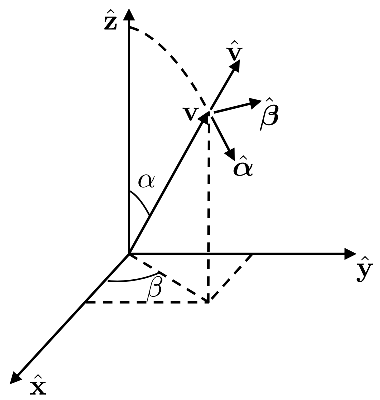

Figure 1: Spherical coordinates v, α and β. The vectors $\hat{\mathbf{v}}$, $\hat{\boldsymbol{\alpha}}$ and $\hat{\boldsymbol{\beta}}$ are the unit vectors parallel to $\nabla_v v$, $\nabla_v \alpha$ and $\nabla_v \beta$.

A simple equation that describes the balance between collisions and injection of momentum and energy due to RF waves is

$$C_{ee}[f_e, f_e] + C_{ei}[f_e, f_i] = -S_{RF}, \tag{1}$$

where $C_{ss'}[f_s, f_{s'}]$ is the Fokker-Planck collision operator between species s and s', and the source S_{RF} models the resonance between electrons and waves with phase velocity $u\hat{\mathbf{z}}$,

$$S_{RF} = -\frac{F_{RF}}{4\pi m_e v^2}\frac{\mathrm{d}}{\mathrm{d}v}[\delta(v-u)] + \frac{3F_{RF}}{4\pi m_e u^3}\delta(v-u)\cos\alpha.$$

Here $\delta(\ldots)$ is the Dirac delta function and F_{RF} is a constant. Assume that the energy and momentum injection rates are sufficiently low that the ion and electron distribution functions are close to stationary Maxwellians,

$$f_s(\mathbf{v}) \simeq f_{Ms}(v) = n_s\left(\frac{m_s}{2\pi T_s}\right)^{3/2}\exp\left(-\frac{m_s v^2}{2T_s}\right),$$

where n_s, T_s and m_s are the density, temperature and mass of species s, respectively.

(a) [5 marks] Show that the source S_{RF} does not affect particle balance and that the momentum input per unit time and per unit volume due to S_{RF} is $F_{RF}\hat{\mathbf{z}}$. Calculate the energy injection per unit volume and per unit time.

(b) [5 marks] Using a moment of the kinetic equation (1) and expanding in $\sqrt{m_e/m_i} \ll 1$, calculate the steady-state temperature difference $T_e - T_i$.

(c) [10 marks] In most applications, the phase velocity of the wave u is large compared to the electron thermal speed $v_{te} = \sqrt{2T_e/m_e}$, that is, $u \gg v_{te}$. At such large speeds, the distribution function is small and the radiofrequency heating can have a large effect, leading to a non-Maxwellian tail; that is, $f_e \not\simeq f_{Me}$ for $v \sim u \gg v_{te}$, but $f_e \simeq f_{Me}$ for $v \sim v_{te}$. Fortunately, we can still simplify the collision operator because the total contribution of the tail to integrals of the distribution function is smaller than the contribution due to the thermal part of the distribution function. Neglecting the contribution from the non-Maxwellian tail to the integrals in the collision operator but keeping its contribution to the differential terms, one finds that $C_{ee}[f_e, f_e] \simeq C_{ee}[f_e, f_{Me}]$ for velocities $v \sim u \gg v_{te}$. To simplify the electron-electron collision operator even further, assume that

$$f_e \sim f_{Me}, \quad \frac{\partial f_e}{\partial \alpha} \sim \frac{\partial f_e}{\partial \beta} \sim f_{Me} \quad \text{and} \quad \frac{\partial f_e}{\partial v} \sim \frac{f_{Me}}{v_{te}}$$

for $v \sim u \gg v_{te}$. Using these assumptions, expand the collision operator in $v_{te}/u \sim v_{te}/v \ll 1$ to find

$$C_{ee}[f_e, f_e] \simeq \frac{3\sqrt{\pi}\nu_{ei}}{8Z}\left(\frac{2T_e}{m_e}\right)^{3/2}\left\{\frac{1}{v^2}\frac{\partial}{\partial v}\left[\frac{2T_e f_{Me}}{m_e v}\frac{\partial}{\partial v}\left(\frac{f_e}{f_{Me}}\right)\right]\right.$$
$$\left. + \frac{1}{v^3}\left[\frac{1}{\sin\alpha}\frac{\partial}{\partial \alpha}\left(\sin\alpha\frac{\partial f_e}{\partial \alpha}\right) + \frac{1}{\sin^2\alpha}\frac{\partial^2 f_e}{\partial \beta^2}\right]\right\},$$

where $\nu_{ei} = (4\sqrt{2\pi}/3)[Ze^4 n_e \ln\Lambda_{ei}/(4\pi\epsilon_0)^2 m_e^{1/2} T_e^{3/2}]$ is the electron-ion collision frequency, $\ln\Lambda_{ei}$ is the Coulomb logarithm, and ϵ_0 is the vacuum permittivity.

[Hint: you may use the expressions for $C_{ss'}[f_s, f_{Ms'}]$ given in the notes/homework.]

(d) [10 marks] The solution to equation (1) can be decomposed into two spherical harmonics,

$$f_e = F_e(v) + G_e(v)\cos\alpha.$$

Find the equation for $F_e(v)$ and discuss the boundary conditions that one must impose on F_e. Show that the solution for $F_e(v)$ is

$$F_e(v) = f_{Me} + K\mathcal{H}(v-u)\exp\left(-\frac{m_e(v^2-u^2)}{2T_e}\right). \tag{2}$$

where K is a constant and $\mathcal{H}(s)$ is the Heaviside step function, that is, $\mathcal{H}(s) = 0$ for $s < 0$ and $\mathcal{H}(s) = 1$ for $s > 0$. Determine the constant K and calculate the effective electron temperature $\Theta_e = n_e^{-1}\int f_e(m_e v^2/3)\,\mathrm{d}^3v$. Compare $\Theta_e - T_e$ with $T_e - T_i$, calculated in part (b), and comment on the result.

[Hint: to calculate Θ_e, you may want to use the integration variable $w = v - u$, and the fact that in the integral $w \sim T_e/m_e u \ll u$.]

(e) [15 marks] The piece G_e satisfies

$$\frac{\mathrm{d}G_e}{\mathrm{d}v} \sim \frac{G_e}{u} \tag{3}$$

for $v \sim u \gg v_{te}$. Show that, due to these orderings, the equation for G_e simplifies to

$$\frac{3\sqrt{\pi}\nu_{ei}}{4Zv^2}\left(\frac{2T_e}{m_e}\right)^{3/2}\left(\frac{\mathrm{d}G_e}{\mathrm{d}v} - \frac{1+Z}{v}G_e\right) = -\frac{3F_{RF}}{4\pi m_e u^3}\delta(v-u).$$

Solve for G_e and calculate the electron flux $n_e\mathbf{u}_e = \int f_e \mathbf{v}\,\mathrm{d}^3v$. Compare the result with the electron flux driven by an electric field force that is of the same size as the momentum injected by the waves, $-en_e\mathbf{E} = F_{RF}\hat{\mathbf{z}}$.

2. In Braginskii's model, the expressions for the parallel electron heat flux and the parallel friction are

$$q_{e\|} = -\frac{A_{11} p_e}{m_e \nu_{ei}} \hat{\mathbf{b}} \cdot \nabla T_e + A_{12} p_e (u_{i\|} - u_{e\|}),$$
$$F_{ei\|} = A_{21} n_e \hat{\mathbf{b}} \cdot \nabla T_e + A_{22} n_e m_e \nu_{ei} (u_{i\|} - u_{e\|}),$$

where n_e, T_e and $p_e = n_e T_e$ are the electron density, temperature and pressure, respectively, $u_{i\|} - u_{e\|}$ is the difference between the ion and electron fluid parallel velocities, $\hat{\mathbf{b}} = \mathbf{B}/B$ is the unit vector in the direction of the magnetic field $\mathbf{B}$, m_e is the electron mass, $\nu_{ei} = (4\sqrt{2\pi}/3)[Ze^4 n_e \ln\Lambda_{ei}/(4\pi\epsilon_0)^2 m_e^{1/2} T_e^{3/2}]$ is the electron-ion collision frequency, e is the proton charge, Ze is the ion charge, $\ln\Lambda_{ei}$ is the Coulomb logarithm, and ϵ_0 is the vacuum permittivity. The coefficients A_{11}, A_{12}, A_{21} and A_{22} are order unity functions of the charge number of the ions Z.

In the notes, we derived the coefficients A_{11}, A_{12}, A_{21} and A_{22} for $Z = 1$. We found that A_{11} and A_{22} are positive, A_{22} is less than one, and $A_{12} = A_{21}$ (the equality $A_{12} = A_{21}$ is known as Onsager symmetry). In this problem, you will prove these properties of the coefficients A_{11}, A_{12}, A_{21} and A_{22} for all Z.

(a) [5 marks] Show that the equation for the gyroaveraged first order perturbation to the electron distribution function, $\langle f_{e1}\rangle_\varphi$, can be written in the form

$$\nu_{ei}\mathcal{C}[\langle f_{e1}\rangle_\varphi] = \left[F_T(w) w_\| \hat{\mathbf{b}} \cdot \nabla \ln T_e + \frac{F_J(w) m_e \nu_{ei} w_\| (u_{i\|} - u_{e\|})}{T_e}\right] f_{Me},$$

where $\mathcal{C}[\langle f_{e1}\rangle_\varphi]$ is a linear integro-differential operator, and $F_T(w)$ and $F_J(w)$ are functions of the velocity magnitude. Give explicitly $\mathcal{C}[f]$, $F_T(w)$ and $F_J(w)$. Show as well that the parallel electron heat flux and the friction can be written as

$$q_{e\|} = T_e \int \langle f_{e1}\rangle_\varphi F_T(w) w_\| \, \mathrm{d}^3 w,$$
$$F_{ei\|} = n_e m_e \nu_{ei}(u_{i\|} - u_{e\|}) + m_e \nu_{ei} \int \langle f_{e1}\rangle_\varphi F_J(w) w_\| \, \mathrm{d}^3 w.$$

(b) [5 marks] Show that, for gyrophase-independent functions f and g that satisfy $\int f \, \mathrm{d}^3 w = 0 = \int g \, \mathrm{d}^3 w$, $\int f w_\| \, \mathrm{d}^3 w = 0 = \int g w_\| \, \mathrm{d}^3 w$ and $\int f w^2 \, \mathrm{d}^3 w = 0 = \int g w^2 \, \mathrm{d}^3 w$, the linear operator $\mathcal{C}[f]$ satisfies the properties

$$\int \frac{f}{f_{Me}} \mathcal{C}[f] \, \mathrm{d}^3 w < 0.$$

and

$$\int \frac{g}{f_{Me}} \mathcal{C}[f] \, \mathrm{d}^3 w = \int \frac{f}{f_{Me}} \mathcal{C}[g] \, \mathrm{d}^3 w.$$

(c) [10 marks] Using parts (a) and (b), prove that $A_{11} > 0$, $A_{22} < 1$ and $A_{12} = A_{21}$.
[Hint: you may want to use the functions $G_T(w)$ and $G_J(w)$ defined such that $\mathcal{C}[G_\alpha w_\| f_{Me}] = F_\alpha w_\| f_{Me}$ with $\alpha = T, J$.]

(d) [10 marks] Set $\hat{\mathbf{b}} \cdot \nabla T_e = 0$ and rewrite the equation for $\langle f_{e1}\rangle_\varphi$ as

$$C_{ee}^{(\ell)}[g_e] + \mathcal{L}_{ei}[g_e] = \frac{F_{ei\|} w_\|}{p_e} f_{Me}, \tag{4}$$

where $C_{ee}^{(\ell)}$ is the linearized electron-electron collision operator, $\mathcal{L}_{ei}$ is the Lorentz operator for electron-ion collisions, $g_e = \langle f_{e1}\rangle_\varphi + K w_\| f_{Me}$ and K is a constant. Determine the constant K. Use $\int g_e w_\| \, \mathrm{d}^3 w$ and equation (4) to show that $A_{22} > 0$.

3. Consider a plasma composed of one ion species with charge e and mass m_i and electrons with charge $-e$ and mass m_e. This plasma sits on a constant magnetic field pointing in the z-direction, $\mathbf{B} = B\hat{\mathbf{z}}$. The magnetic field is sufficiently large and collisions are sufficiently frequent that the use of Braginskii's equations is justified. The density n_e and the temperatures T_i and T_e are spatially uniform, and the plasma current density $\mathbf{J}$ is zero. The ion velocity $\mathbf{u}_i(t,x) = u_{iz}(t,x)\hat{\mathbf{z}}$ is a function of x and time, and it points in the z-direction.

   (a) [10 marks] Show that the total-momentum equation can be written as

   $$\frac{\partial u_{iz}}{\partial t} = \frac{6 m_i T_i \nu_{ii}}{5e^2B^2}\frac{\partial^2 u_{iz}}{\partial x^2},$$

   where $\nu_{ii} = (4\sqrt{\pi}/3)[e^4 n_e \ln\Lambda_{ii}/(4\pi\epsilon_0)^2 m_i^{1/2} T_i^{3/2}]$ is the ion-ion collision frequency, $\ln\Lambda_{ii}$ is the Coulomb logarithm, and ϵ_0 is the vacuum permittivity.

   (b) [10 marks] Solve the equation for $u_{iz}(t,x)$ for the initial condition

   $$u_{iz}(t=0,x) = \begin{cases} -u_0 & \text{for } x < 0, \\ u_0 & \text{for } x > 0. \end{cases}$$

   and the boundary conditions $u_{iz}(t, x \to -\infty) = -u_0$ and $u_{iz}(t, x \to \infty) = u_0$. Sketch the time evolution of u_{iz}.
   [Hint: you may want to use the self-similar form $u_{iz}(t,x) = U(\xi)$ with $\xi = x/\sqrt{t}$.]

   (c) [5 marks] What difference, if any, does it make if the ion velocity is pointing in the y direction; i.e. $\mathbf{u}_i(t,x) = u_{iy}(t,x)\hat{\mathbf{y}}$?