Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

COLLISIONAL PLASMA PHYSICS

Trinity Term 2017

TUESDAY, 20 JUNE 2017, 12noon to THURSDAY 22 JUNE 2017, 12noon

You should submit answers to ***all*** *questions.*
Answer booklets are provided for you to use but you may type your answers if you wish. Typed answers should be printed single-sided and the pages securely fastened together.

You may refer to books and other sources when completing the exam but should not discuss the exam with anyone else.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. In this problem, we study the evolution of a beam of particles of species α with charge $Z_\alpha e$ and mass m_α that collides with a background composed of particles of species β with charge $Z_\beta e$ and mass m_β. At $t = 0$, the distribution function of particles of species α is

$$f_\alpha(\mathbf{v}, t = 0) = n_{\alpha 0}\delta(\mathbf{v} - u_{\alpha 0}\hat{\mathbf{z}}), \tag{1}$$

where $n_{\alpha 0}$ and $u_{\alpha 0}$ are the density and average velocity of particles of species α at $t = 0$, and $\delta(\mathbf{v})$ is the 3D Dirac delta function. The distribution function of species β is a stationary-Maxwellian,

$$f_\beta(v) = n_\beta \left(\frac{m_\beta}{2\pi T_\beta}\right)^{3/2} \exp\left(-\frac{m_\beta v^2}{2T_\beta}\right). \tag{2}$$

Here $v = |\mathbf{v}|$ and n_β and T_β are the density and temperature of species β. The density n_α is sufficiently low that we can ignore the collisions between particles of species α and we only need to consider collisions of particles of species α with particles of species β, leading to the linear kinetic equation

$$\frac{\partial f_\alpha}{\partial t} = C_{\alpha\beta}[f_\alpha, f_\beta]. \tag{3}$$

Here $C_{\alpha\beta}$ is the Fokker-Planck collision operator.

(a) [10 marks] Derive the following form of the collision operator:

$$C_{\alpha\beta}[f_\alpha, f_\beta] = \nabla_v \cdot \left[f_{M\alpha}(v) \left(\frac{\nu_\parallel(v)}{2}\mathbf{vv} + \frac{\nu_\perp(v)}{4}(v^2\mathbf{I} - \mathbf{vv}) \right) \cdot \nabla_v \left(\frac{f_\alpha}{f_{M\alpha}(v)} \right) \right]. \tag{4}$$

Here $f_{M\alpha}(v)$ is a stationary-Maxwellian with the temperature of species β, T_β, and the density of species α, $n_\alpha = \int f_\alpha \, \mathrm{d}^3 v$. (You may refer to the notes and/or the homework to avoid taking the integrals that lead to $\nu_\perp(v)$ and $\nu_\parallel(v)$.)

(b) [10 marks] To study the behavior of f_β for short times t, $\nu_\perp t \sim \nu_\parallel t \ll 1$, we use the cylindrical coordinates $w_z = v_z - u_{\alpha 0}$, $w_\perp^2 = v_x^2 + v_y^2$ and $\varphi = \arctan(v_y/v_x)$. Assuming $w_x/u_{\alpha 0} \sim w_\perp/u_{\alpha 0} \sim \sqrt{\nu_\perp t} \sim \sqrt{\nu_\parallel t} \ll 1$ and $\varphi \sim 1$, show that the equation for $\nu_\perp t \sim \nu_\parallel t \ll 1$ is

$$\frac{\partial f_\alpha}{\partial t} \simeq \frac{1}{w_z^p}\frac{\partial}{\partial w_z}\left(w_z^p D_\parallel \frac{\partial f_\alpha}{\partial w_z} \right) + \frac{1}{w_\perp^q}\frac{\partial}{\partial w_\perp}\left(w_\perp^q D_\perp \frac{\partial f_\alpha}{\partial w_\perp} \right) + \frac{D_\perp}{w_\perp^r}\frac{\partial^2 f_\alpha}{\partial \varphi^2}. \tag{5}$$

Determine the exponents p, q and r and the constants $D_\parallel$ and $D_\perp$. Give explicitly the dependence of the constants $D_\parallel$ and $D_\perp$ on $u_{\alpha 0}$.

(c) [5 marks] Show (by substitution or otherwise) that the solution to equation (5) with initial condition (1) is of the form

$$f_\alpha(w_z, w_\perp, t) = \frac{N}{t^{3/2}} \exp\left(-\frac{\tau_\parallel w_z^2}{2u_{\alpha 0}^2 t} - \frac{\tau_\perp w_\perp^2}{2u_{\alpha 0}^2 t} \right). \tag{6}$$

Determine the constants N, $\tau_\parallel$ and $\tau_\perp$ and describe the evolution of f_α. Are our assumptions $w_z/u_{\alpha 0} \sim w_\perp/u_{\alpha 0} \sim \sqrt{\nu_\perp t} \sim \sqrt{\nu_\parallel t} \ll 1$ consistent with this solution?

2. Question 1 gives the short time evolution of the beam. In this question, we study the long time behavior, $\nu_\perp t \sim \nu_\parallel t \gg 1$.

(a) [5 marks] Rewrite the collision operator in (4) using the spherical coordinates $\{v, \theta, \varphi\}$ shown in figure 1.

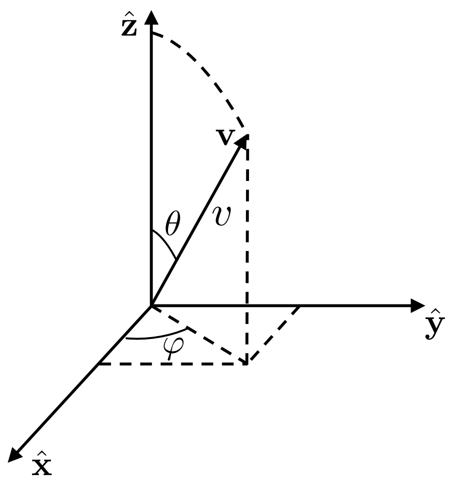

Figure 1: Spherical coordinates $\{v, \theta, \varphi\}$ for the velocity $\mathbf{v}$.

(b) [5 marks] To solve the equation derived in part (a), we decompose f_α into the series

$$f_\alpha(v, \theta, \varphi, t) = f_{M\alpha}(v) \sum_{n=0}^{\infty} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} A_{nlm} \exp(-\overline{\nu}_{nl} t) F_{nl}(v) Y_l^m(\theta, \varphi). \tag{7}$$

Here A_{nlm} and $\overline{\nu}_{nl}$ are constants, the functions $Y_l^m(\theta, \varphi)$ are spherical harmonics, and the functions $F_{nl}(v)$ have to be determined. The function F_{nl} vanishes for $v \to \infty$ and it is regular at $v = 0$. Show that $\overline{\nu}_{nl}$ and $F_{nl}(v)$ are determined by the eigenvalue problem

$$\frac{1}{v^2 f_{M\alpha}} \frac{\mathrm{d}}{\mathrm{d}v} \left(\frac{v^4 \nu_{\|} f_{M\alpha}}{2} \frac{\mathrm{d}F_{nl}}{\mathrm{d}v} \right) + \left[\overline{\nu}_{nl} - \frac{l(l+1)\nu_{\perp}}{4} \right] F_{nl} = 0. \tag{8}$$

(c) [10 marks] Show that the eigenvalues $\overline{\nu}_{nl}$ are extrema of the functional

$$\Theta[F] = \frac{1}{\int_0^\infty v^2 f_{M\alpha} F^2 \, \mathrm{d}v} \left[\frac{1}{2} \int_0^\infty \nu_{\|} v^4 f_{M\alpha} \left(\frac{\mathrm{d}F}{\mathrm{d}v} \right)^2 \mathrm{d}v + \frac{l(l+1)}{4} \int_0^\infty \nu_{\perp} v^2 f_{M\alpha} F^2 \, \mathrm{d}v \right]. \tag{9}$$

In other words, show that $\Theta[F_{nl}] = \overline{\nu}_{nl}$ and that $\Theta[F_{nl} + \delta F] - \Theta[F_{nl}] = 0$ to first order in $\delta F \ll F_{nl}$.

(d) [5 marks] Using the variational principle in equation (9), one can estimate the eigenvalues $\overline{\nu}_{nl}$ for different l, and as a result, one can find the exponential decay of the solution at $t \to \infty$. In this exam, we are only going to obtain the approximate value of the smallest decay rate $\overline{\nu}_{00}$ for $l = 0$. Using the trial function $F_{00}(v, K) = m_\alpha v^2 / 2T_\beta - K$, show that

$$\Theta_{00}(K) \equiv \Theta[F_{00}(v, K)] = \frac{N}{K^2 + aK + b}. \tag{10}$$

Determine the constants N, a and b. Then, by minimizing $\Theta_{00}(K)$ with respect to K, determine the smallest decay rate for $l = 0$.

[Hint:

$$\int_0^\infty \nu_{\|} v^6 f_{M\alpha} \, \mathrm{d}v = \frac{3}{4\pi\sqrt{2}} \frac{n_\alpha T_\beta^2 \nu_{\alpha\beta}}{m_\alpha^2}, \tag{11}$$

where

$$\nu_{\alpha\beta} = \frac{16\sqrt{\pi}}{3} \frac{Z_\alpha^2 Z_\beta^2 e^4 n_\beta m_\alpha^{1/2} m_\beta^{1/2} \ln \Lambda_{\alpha\beta}}{(4\pi\epsilon_0)^2 (m_\alpha + m_\beta)^{3/2} T_\beta^{3/2}} \tag{12}$$

is the frequency of collisions of particles of species α with particles of species β.]

3. Consider a plasma composed of one ion species with charge e and mass m_i, and electrons with charge $-e$ and mass m_e in a gravitational field $-g\hat{\mathbf{z}}$ (that is, for species $s = i, e$, in addition to the pressure force $-\nabla p_s$ and the electromagnetic force $Z_s e(\mathbf{E} + \mathbf{u}_s \times \mathbf{B})$, we need to include the force $-n_s m_s g\hat{\mathbf{z}}$ in the momentum equations). The plasma satisfies Braginskii's assumptions

$$\frac{\rho_e}{L} \ll \frac{\rho_i}{L} \ll \frac{\lambda_{ii}}{L} \sim \frac{\lambda_{ee}}{L} \sim \frac{\lambda_{ei}}{L} \ll 1, \tag{13}$$

where ρ_s is the characteristic gyroradius of species s, and $\lambda_{ss'}$ is the mean free path for collisions between particles of species s and particles of species s'. In addition to Braginskii's assumptions, the plasma satisfies

$$\frac{\rho_i}{L} \ll \frac{v_A}{v_{ti}} \ll \frac{|\mathbf{u}_s|}{v_{ti}} \sim \frac{\sqrt{gL}}{v_{ti}} \sim \frac{\lambda_{ii}}{L}\sqrt{\frac{m_i}{m_e}} \ll 1, \tag{14}$$

where L is the characteristic size of the system, $v_A = B/\sqrt{n_i m_i \mu_0}$ is the Alfven speed, and n_s, $\mathbf{u}_s$, T_s and $v_{ts} = \sqrt{2T_s/m_s}$ are the density, average velocity, temperature and thermal speed of species s, respectively.

(a) [10 marks] Show that the lowest-order total-momentum equation gives $\nabla p \simeq 0$, where $p = n_i T_i + n_e T_e$ is the total pressure (note that neither n_s nor T_s must have small gradients). Show as well that Ampere's law gives $\mathbf{u}_e \simeq \mathbf{u}_i \equiv \mathbf{u}$, and that the lowest-order electron-energy equation gives $T_i \simeq T_e \equiv T$. Thus, using quasineutrality $n_i = n_e \equiv n$, the lowest-order total-momentum equation becomes

$$\nabla(nT) = 0. \tag{15}$$

(b) [5 marks] Show that the lowest-order vorticity equation (curl of the total-momentum equation) is

$$\nabla \times \left[n m_i \left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) \right] = m_i g \hat{\mathbf{z}} \times \nabla n, \tag{16}$$

(c) [10 marks] Show that the lowest-order total-energy equation can be written as

$$nT \left(\frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla \right) \ln \left(\frac{T}{n^{2/3}} \right) - \nabla \cdot \left(\frac{1.05 nT}{m_e \nu_{ee}} \hat{\mathbf{b}}\hat{\mathbf{b}} \cdot \nabla T \right) = 0. \tag{17}$$

[Hint: use the continuity equation to eliminate $\nabla \cdot \mathbf{u}$ from the total-energy equation].

(d) [5 marks] Using the electron-momentum equation, show that

$$\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}). \tag{18}$$

4. A possible steady-state solution to the equations derived in question 3 is $\mathbf{u} = 0$, a constant magnetic field $\mathbf{B} = B\hat{\mathbf{x}}$, and density and temperature $n(z)$ and $T(z)$ that only depend on z and satisfy $n(z)T(z) =$ constant. In this question we are going to study the stability of this solution.

(a) [15 marks] Linearize the equations derived in question 3 around the steady-state solution proposed in this question. Use perturbations of the form $Q_1 = \widetilde{Q}\exp(\gamma t + ikx)$, with $kL \gg 1$, and assume that $\mathbf{u}_1 \cdot \hat{\mathbf{y}} = 0 = \mathbf{B}_1 \cdot \hat{\mathbf{y}}$. You should find the final equations

$$\frac{\tilde{n}}{n} + \frac{\tilde{T}}{T} = 0, \tag{19}$$

$$\tilde{u}_z = -\frac{g}{\gamma}\frac{\tilde{n}}{n}, \tag{20}$$

$$\frac{\tilde{T}}{T} - \frac{2}{3}\frac{\tilde{n}}{n} + \frac{\tilde{u}_z}{\gamma}\frac{\mathrm{d}}{\mathrm{d}z}\ln\left(\frac{T}{n^{2/3}}\right) + \frac{1.05T}{\gamma m_e \nu_{ee}}\left(k^2\frac{\tilde{T}}{T} - \frac{ik\tilde{B}_z}{B}\frac{\mathrm{d}\ln T}{\mathrm{d}z}\right) = 0, \tag{21}$$

$$\frac{\tilde{B}_z}{B} = \frac{ik\tilde{u}_z}{\gamma}. \tag{22}$$

[Hint: you only need to use one component of the vorticity equation (16) and one component of the induction equation (18).]

(b) [5 marks] Find γ in the limit $\gamma \ll k^2 T/m_e\nu_{ee}$. What is the stability condition in this limit? (Note that this mode is known as MagnetoThermal Instability, and it is important in astrophysical plasmas. It is driven by the fact that in a magnetized plasma, the heat flux is parallel to the magnetic field lines.)