A15757W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

ADVANCED QUANTUM THEORY

Trinity Term 2022

Thursday 9th June 2022, 9:30am to 11:30am

You should submit answers to both questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

You are permitted to use the following material(s):
A4 summary sheet

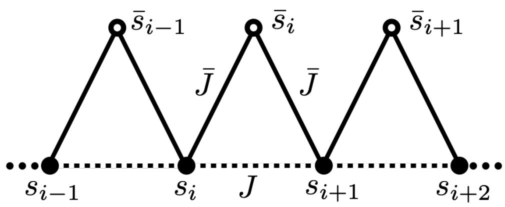
The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. [25 marks] The classical sawtooth Ising model, depicted in the figure below, consists of a chain of corner-sharing triangles hosting two sets of Ising spins: s_i on the base of the chain and $\bar{s}_i$ on the apex of each triangle, with $s_i, \bar{s}_i \in \{-1, +1\}$. We assume there are L unit cells with periodic boundary conditions, so that $s_L \equiv s_0$ and $\bar{s}_0 \equiv \bar{s}_L$. The spins interact via *antiferromagnetic* exchange couplings $\bar{J} \geqslant J > 0$ as shown in the figure, so that the classical energy of the chain is given by

$$E[\{s_i, \bar{s}_i\}] = \sum_{i=0}^{L-1} \left[J s_i s_{i+1} + \bar{J} s_i \bar{s}_i + \bar{J} \bar{s}_i s_{i+1}\right]. \tag{1}$$

Note that there is no direct interaction between apex spins, and that each unit cell contains two spins.



(a) [7 marks] Show that the partition function of the system at inverse temperature $\beta \equiv 1/k_B T$ can be written in the form

$$Z[\beta, J, \bar{J}] = \mathrm{Tr}\left\{(A[\beta, J, \bar{J}])^L\right\}, \tag{2}$$

where $A[\beta, J, \bar{J}]$ is a 2×2 transfer matrix whose components are given by

$$A[\beta, J, \bar{J}]_{ss'} = e^{-\beta J s s'} 2\cosh[\beta \bar{J}(s + s')]. \tag{3}$$

Your derivation should clearly demonstrate why the configurations of apex spins do not appear explicitly as indices of the transfer matrix.

(b) [5 marks] Compute the free energy density (defined as the free energy per *spin*) as a function of $J, \bar{J}$, and β in the thermodynamic limit, $L \to \infty$.

(c) [7 marks] Determine the behaviour of the entropy per spin as $T \to 0$. Contrast its behaviour when $\bar{J} = J$ and when $\bar{J} > J$. Also determine the high-temperature ($T \to \infty$) behaviour of this quantity, and explain why it is independent of $\bar{J}$ and J.

(d) [6 marks] Determine the minimum-energy configurations of a single triangle when $\bar{J} = J$ and when $\bar{J} > J$. By considering how placing spins in their minimum-energy configuration on one triangle influences adjacent triangles, give a qualitative explanation for the results of part (c). Does a similar distinction between $\bar{J} = J$ and $\bar{J} > J$ arise when the interactions are *ferromagnetic*?

2. [25 marks] Consider the one-dimensional spin chain described by the Hamiltonian

$$H = \sum_i \left(-J[\boldsymbol{S}_i \cdot \boldsymbol{S}_{i+1} + u(\hat{\mathbf{d}}_1 \cdot \boldsymbol{S}_i)(\hat{\mathbf{d}}_1 \cdot \boldsymbol{S}_{i+1})] - D\hat{\mathbf{d}}_2 \cdot \boldsymbol{S}_i \times \boldsymbol{S}_{i+1} \right), \tag{1}$$

where $\boldsymbol{S}_i = (S_i^x, S_i^y, S_i^z)$ are spin-S operators (we assume $S \gg 1/2$), $\hat{\mathbf{d}}_1$ and $\hat{\mathbf{d}}_2$ are fixed unit vectors, and we have the standard commutation relations $[S_l^\alpha, S_m^\beta] = i\delta_{lm}\epsilon_{\alpha\beta\gamma}S_l^\gamma$. We assume the chain has L sites and periodic boundary conditions, so that $\boldsymbol{S}_0 \equiv \boldsymbol{S}_L$. Throughout, we will take $J > D \geqslant 0$, and $u \geqslant -1$.

For the first part of this problem, set $\hat{\mathbf{d}}_1 = \hat{\mathbf{d}}_2 = \hat{\mathbf{z}}$. In this case, H is invariant under rotations in spin space about $\hat{\mathbf{z}}$, and under the Ising symmetry operation $S_i^z \to -S_i^z$.

(a) [5 marks] The Holstein-Primakoff representation is defined by

$$S_j^z = S - a_j^\dagger a_j, \qquad S_j^+ = S_j^x + iS_j^y = (2S - a_j^\dagger a_j)^{1/2} a_j, \qquad [a_j, a_l^\dagger] = \delta_{j,l}. \tag{2}$$

Explain the nature and usefulness of this representation, taking care to discuss the role of constraints in making the mapping from spins to bosons meaningful. Give any two complications that may arise with using this representation.

(b) [6 marks] For the case $u \gg 1$, using the Holstein-Primakoff approach, carry out an expansion of H in inverse powers of S. Ignore the constant contribution and drop all terms that grow more slowly than S when S becomes large. Show that the resulting Hamitonian H_{LSW}, the linear spin wave (LSW) approximation to H, takes the form

$$H_{LSW} = \sum_{j=1}^{L} (A a_{j+1}^\dagger a_j + A^* a_j^\dagger a_{j+1}) + B a_j^\dagger a_j, \tag{3}$$

where A and B are constants. Determine the value of these constants.

(c) [6 marks] Show that H_{LSW} can be written in the form

$$H_{LSW} = \sum_k \epsilon(k) b^\dagger(k) b(k), \tag{4}$$

where $b^\dagger(k), b(k)$ are bosonic creation and annihilation operators, and the sum is over the discrete set of momenta $k = \frac{2\pi j}{L}$, $j = 0, 1, \ldots, L-1$. What is the ground state $|\mathrm{GS}\rangle$ of H_{LSW}? What are the low-lying excitations and what are their energies?

(d) [4 marks] Show that the energy of the lowest-lying excited state computed from H_{LSW} becomes equal to that of the ground state at $u = u_c$, where $u_c > 0$ is a constant whose value you should determine. This 'energy gap closing' signals a phase transition at $u = u_c$.

(e) [4 marks] Now, consider a more general situation where $\hat{\mathbf{d}}_1$ and $\hat{\mathbf{d}}_2$ are arbitrary. When $u = D = 0$, H reduces to the Heisenberg model, which is invariant under arbitrary rotations (you need not prove this). By an explicit calculation, show that when $D = 0$, H is invariant under spin rotations about $\hat{\mathbf{d}}_1$, whereas when $u = 0$, H is invariant under spin rotations about $\hat{\mathbf{d}}_2$. Hence conclude that if both D and u are nonzero, H is not invariant under spin rotations unless $\hat{\mathbf{d}}_1 = \hat{\mathbf{d}}_2$, as in the first part of the problem. [*Hint: determine the conditions for the vanishing of commutators of the form* $[H, \hat{\boldsymbol{n}} \cdot \boldsymbol{S}^{\mathrm{tot}}]$ *where* $\boldsymbol{S}_i^{\mathrm{tot}} = \sum_j \boldsymbol{S}_j$, *using translational invariance wherever possible.*]