A15757W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

ADVANCED QUANTUM THEORY: PATH INTEGRALS AND MANY-PARTICLE PHYSICS

Trinity Term 2021

WEDNESDAY, 9TH JUNE 2021, Opening Time 09:30 am UK Time

You should submit answers to both questions.

You have **2 hours** *writing time to complete the paper and up to* **30 minutes** *technical time for uploading your file. The allotted technical time must not be used to finish writing the paper.*
Mode of completion (format in which you will complete this exam): **handwritten**
You are permitted to use the following material(s):
Calculator (candidate to provide)
The use of computer algebra packages is **not** *allowed.*

1. A one-dimensional simple harmonic oscillator has mass m and frequency ω. [25]

   (a) Write down the path integral for the propagator $\langle x|U(t;0)|x'\rangle$, specifying any conditions on the paths involved in this expression. [3]

   (b) Write down the equations of motion obeyed by a classical path $x_{\rm cl}$. Show that the propagator can be written in the form [6]

$$\langle x|U(t;0)|x'\rangle = A(t)e^{\frac{i}{\hbar}S_{\rm cl}[x,x',t]}, \qquad (1)$$

   where $S_{\rm cl}[x,x',t]$ is the action for a classical path $x_{\rm cl}$ that starts at position x' at time $t=0$ and ends at position x at time t, and

$$A(t) = \int \mathcal{D}y \, \exp\left[\frac{i}{\hbar}\int_0^t dt' \, \frac{m}{2}\left\{\left(\frac{dy}{dt'}\right)^2 - \omega^2 y^2\right\}\right] \qquad (2)$$

   is a path integral over paths satisfying $y(0) = y(t) = 0$, and is therefore independent of x and x'. (*Hint: express the paths in the path integral as* $x(t) = x_{\rm cl}(t) + y(t)$ *and use the equations of motion obeyed by* $x_{\rm cl}$.)

   (c) By considering the most general solution to the classical equations of motion and imposing appropriate boundary conditions, show that [6]

$$S_{\rm cl}[x,x',t] = \frac{m\omega}{2\sin\omega t}\left[(x^2+x'^2)\cos\omega t - 2xx'\right]. \qquad (3)$$

   (d) Using the results of parts (b) and (c) and the fact that the ground state wave-function [8]

$$\psi_0(x) = \left(\frac{m\omega}{\sqrt{\pi}\hbar}\right)^{1/2} e^{-m\omega x^2/2\hbar} \qquad (4)$$

   must be an eigenstate of the time evolution operator, deduce an expression for $A(t)$ in terms of m, ω, t, and $\hbar$. (*Hint: start from the eigenvalue equation* $U(t;0)|\psi_0\rangle = e^{-iE_0t/\hbar}|\psi_0\rangle$ *with* $E_0 = \hbar\omega/2$.)

   (e) Combining the results above, write down an explicit expression for the path integral of the harmonic oscillator $\langle x|U(t;0)|x'\rangle$ in terms of x, x', t, m, ω, and $\hbar$. Check your answer by verifying that in the limit $\omega \to 0$, it reduces to the free-particle propagator, [2]

$$\langle x|U_{\rm free}(t;0)|x'\rangle = \sqrt{\frac{m}{2\pi i\hbar t}}e^{\frac{im}{2\hbar t}(x-x')^2}. \qquad (5)$$

You may find the following formula useful: if $\mathrm{Re}(z) > 0$,

$$\int_{-\infty}^{\infty} dy \, e^{-\frac{1}{2}zy^2+Jy} = \sqrt{\frac{2\pi}{z}}e^{J^2/2z}.$$

2. In some systems, the low-temperature ordered phase changes from spatially uniform (the "ferromagnet", denoted F) to spatially modulated (e.g. with $\phi(x)$ periodic in space, denoted M), as a function of some experimentally tunable parameter. Including the high-temperature disordered phase (the "paramagnet", P), there are three distinct phase boundaries, P-F, P-M, and F-M, which meet at a *Lifshitz point*, LP. A transition from a spatially uniform to a periodically modulated phase is first-order if the period becomes nonzero discontinuously across the transition. [25]

A Landau-Ginzburg theory that captures this behaviour for a real or complex scalar order parameter ϕ in one spatial dimension takes the form

$$\beta\mathcal{H} = \int dx \left[\alpha_2(T)|\phi|^2 + \rho_2|\partial_x\phi|^2 + \rho_4|\partial_x^2\phi|^2 + \alpha_4|\phi|^4\right], \tag{6}$$

where α_2 changes sign from positive to negative as the temperature is lowered, and $\alpha_4 > 0$ as usual, but we allow ρ_2 to be either negative or positive as some external parameter is varied. We ignore any temperature dependence in α_4, ρ_2, and ρ_4.

(a) Explain why we can ignore ρ_4 when $\rho_2 > 0$, but not when $\rho_2 < 0$. [2]

(b) When ϕ is a *complex* scalar, consider a mean-field *ansatz* in which a single Fourier mode $\phi(x) = A_k e^{ikx}$ with $k = \pm k_0$ is non-vanishing. Show that it has a free energy density that depends on the choice of k_0: [12]

$$f = (\alpha_2 + \rho_2 k_0^2 + \rho_4 k_0^4)|A_k|^2 + \alpha_4|A_k|^4. \tag{7}$$

By solving the saddle-point equations, determine the values of k_0 and A_k that minimize f, and hence compute the free energy densities f_P, f_F, f_M of the three phases, as functions of $\alpha_2, \rho_2, \rho_4 > 0$ and $\alpha_4 > 0$. Use these to determine the phase boundaries in the $\rho_2 - \alpha_2$ plane, identifying the order of each transition. Sketch the phase diagram.

(c) The Lifshitz point LP is at $\alpha_2 = \rho_2 = 0$. Using the Gaussian approximation and assuming that $\alpha_2(T) = At$ where $t = (T - T_c)/T_c$, determine the mean-field correlation length exponent as this point is approached from phase P along the line $\rho_2 = 0$. (You should not need to perform any integrals explicitly.) [4] [4]

When ϕ is a *real* scalar, we must consider a modified mean-field *ansatz* $\phi(x) = A_k \cos(kx)$ with $k = \pm k_0$. In this case, the F-M phase boundary changes its location and its order; the other phase boundaries are unchanged. Therefore, for the remainder of this question, you can focus on $\alpha_2 < 0$.

(d) For $k_0 \neq 0$, (7) is no longer valid. Determine the free energy density for $k_0 \neq 0$, and solve the resulting saddle-point equations to determine the modified free energy density $\tilde{f}_M$. (*Hint: it may help to divide the system up into unit cells of length* $2\pi/k_0$.) [4]

(e) The free energy f_F of the ferromagnet (with $k_0 = 0$) is unchanged from part (b). By comparing f_F and $\tilde{f}_M$ for $\alpha_2 < 0$ as ρ_2 is varied, determine the new F-M phase boundary and the order of the transition. [3]

You may find the following integrals useful:

$$\int_0^{2\pi} \sin^2 x\, dx = \int_0^{2\pi} \cos^2 x\, dx = \pi, \quad \int_0^{2\pi} \sin^4 x\, dx = \int_0^{2\pi} \cos^4 x\, dx = \frac{3\pi}{4}.$$