A15757W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

ADVANCED QUANTUM THEORY: PATH INTEGRALS AND MANY-PARTICLE PHYSICS

Trinity Term 2019

WEDNESDAY, 19 JUNE 2019, 14:30 to 16:30

*You should submit answers to **any two** out of the three questions.*

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. Consider a near-critical three-dimensional system whose phases may be characterized by two different real scalar order parameters ϕ_1, ϕ_2. The model has two distinct symmetries $\hat{S}_1$, $\hat{S}_2$, that act on the two order parameters as follows: [25]

$$\hat{S}_1 : (\phi_1, \phi_2) \mapsto (-\phi_1, \phi_2); \qquad \hat{S}_2 : (\phi_1, \phi_2) \mapsto (\phi_1, -\phi_2) \tag{1}$$

In the absence of any coupling between the order parameters and of external fields that couple to either order parameter, the Landau free energy can be written as

$$\beta F_0 = \sum_{i=1,2} \int d^3\mathbf{r} \left[\frac{1}{2} |\nabla \phi_i|^2 + t_i \phi_i^2 + u_i \phi_i^4 + \ldots \right] \tag{2}$$

(a) Show that the lowest-order term that can be added to (2) and that couples the two order parameters is [4]

$$\beta(\delta F) = 2g \int d^3\mathbf{r}\ \phi_1^2 \phi_2^2, \tag{3}$$

where the factor of 2 has been inserted for convenience, and $g > 0$ is a constant. Explain why it is sufficient to modify the Landau free energy by keeping only this coupling term, and also why it is reasonable to ignore derivative terms that couple ϕ_1 and ϕ_2.

For parts (b-d) of this question, use the interacting theory with $\beta F = \beta F_0 + \beta(\delta F)$.

(b) Before specifying the values of any of the parameters, this system apparently has four possible phases. Explain what these are in terms of the behaviour of ϕ_1 and ϕ_2. [4]

(c) By determining the minima of the free energy (making any necessary assumptions about t_1, t_2), show that one of the four phases in part (b) is absent if g exceeds a critical value g_c which depends only on u_1, u_2. Determine g_c. [8]

(d) Using results of (b) and (c), sketch the phase diagram of the system in the t_1, t_2 plane for the case $g > g_c$, identifying the order of each transition. [7]

(e) Now, suppose the two order parameters transform under a single symmetry, i.e. $\hat{S} : (\phi_1, \phi_2) \mapsto (-\phi_1, -\phi_2)$. What is the leading term that couples the order parameters in this case? [2]

2. A one-dimensional quantum spin model has the Hamiltonian [25]

$$H = -J \sum_{j=1}^{L} \left(S_j^x S_{j+1}^x + S_j^y S_{j+1}^y + \Delta S_j^z S_{j+1}^z \right) \tag{1}$$

where $J, \Delta > 0$, and we have a spin-S on each site of the lattice, i.e. $(S_j^x)^2 + (S_j^y)^2 + (S_j^z)^2 = S(S+1)$. We impose periodic boundary conditions, $S_{L+1}^\alpha = S_1^\alpha$ for $\alpha = x, y, z$.

(a) For general values of Δ, what symmetry does this model have under simultaneous rotations of all the spins? How does this change for $\Delta = 1$? [You may state these answers without proof.] [4]

(b) What is the classical ground state of this model? Justify your answer by a calculation. Explain how the classical ground state changes depending on whether $\Delta > 1$ or $\Delta < 1$. [*Hint: Consider* (S_j^x, S_j^y, S_j^z) *to be a classical vector of length* S.] [6]

For the remainder of this problem, take $\Delta = 1$.

(c) The Holstein-Primakoff representation is defined by [4]

$$S_j^z = S - a_j^\dagger a_j, \qquad S_j^+ = S_j^x + iS_j^y = \left(2S - a_j^\dagger a_j\right)^{1/2} a_j, \qquad [a_j, a_\ell^\dagger] = \delta_{j,\ell}$$

Explain the nature and usefulness of this representation. Comment on complications that generally could arise.

(d) Using the Holstein-Primakoff approach, carry out an expansion of H (with $\Delta = 1$) in inverse powers of S. Ignore the constant contribution and drop all terms that grow more slowly than S when S becomes large. Show that the resulting Hamitonian H_{LSW}, the linear spin wave (LSW) approximation to H, takes the form [5]

$$H_{\text{LSW}} = \sum_{j=1}^{L} A(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j) + B(a_j a_{j+1} + a_j^\dagger a_{j+1}^\dagger) + C a_j^\dagger a_j$$

where A, B, C are constants. Determine the values of these constants.

(e) Show that H_{LSW} can be written in the form [6]

$$H_{\text{LSW}} = \sum_k \epsilon(k) b^\dagger(k) b(k) + \text{const.},$$

where $b^\dagger(k), b(k)$ are bosonic creation and annihilation operators, and 'const.' denotes a constant contribution that you need not determine. What is the ground state $|GS\rangle$ of H_{LSW}? What are the low-lying excitations and what are their energies?

3. Consider the anharmonic oscillator, whose Hamiltonian is [25]

$$H(\gamma, \lambda) = \frac{\hat{p}^2}{2m} + \frac{\kappa}{2}\hat{x}^2 + \frac{\gamma}{3!}\hat{x}^3 + \frac{\lambda}{4!}\hat{x}^4. \tag{1}$$

where $\kappa, \gamma, \lambda > 0$ are constants, and $\gamma^2 < 3\kappa\lambda$.

(a) What is the path integral representation for the partition function $Z(\beta)$? What kinds of paths are integrated over in the path integral? [4]

(b) Recall that the generating functional for $H(0,0)$ is defined as [5]

$$W_{0,0}[J] \equiv \mathcal{N} \int \mathcal{D}x(\tau)\, \exp\left(-\frac{1}{2}\int_0^{\hbar\beta} d\tau\, \left[x(\tau)\hat{D}x(\tau) - 2J(\tau)x(\tau)\right]\right),$$

where $\hat{D} = -\frac{m}{\hbar}\frac{d^2}{d\tau^2} + \frac{\kappa}{\hbar}$, and $\mathcal{N}$ is an overall normalization constant. Show that this can be expressed in the form

$$W_{0,0}[J] = W_{0,0}[0]\exp\left[\frac{1}{2}\int d\tau d\tau'\, J(\tau)G(\tau-\tau')J(\tau')\right],$$

and hence write down the differential equation and boundary conditions that together determine the 'bare' Green's function $G(\tau)$.

(c) Give the definition of the generating functional $W_{\gamma,\lambda}[J]$ for $H(\gamma,\lambda)$ and argue that [3]

$$W_{\gamma,\lambda}[J] = \exp\left(-\int_0^{\hbar\beta} d\tau' \left\{\frac{\gamma}{3!\hbar}\left[\frac{\delta}{\delta J(\tau')}\right]^3 + \frac{\lambda}{4!\hbar}\left[\frac{\delta}{\delta J(\tau')}\right]^4\right\}\right) W_{0,0}[J].$$

(d) Express the first order in γ and λ corrections to the partition function in terms of the bare Green's function G. Sketch the corresponding Feynman diagrams. [6]

(e) Draw the diagrams describing contributions to the corrections at second order in γ and second order in λ respectively. You may neglect combinatorial factors. [5]

(f) Now, consider the corrections up to second order in γ and λ to the two-point function $\langle T_\tau \hat{x}(\tau_1)\hat{x}(\tau_2)\rangle_\beta$. *Qualitatively* explain the distinction between the cases when $\gamma = 0$ and $\gamma \neq 0$, making reference to the Feynman-diagram representation of the corrections and going up to second order. [2]