Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

ADVANCED QUANTUM THEORY: PATH INTEGRALS AND MANY-PARTICLE PHYSICS

Trinity Term 2017

TUESDAY, 13th JUNE 2017, 2:30pm to 4:30pm

You should submit answers to both questions.

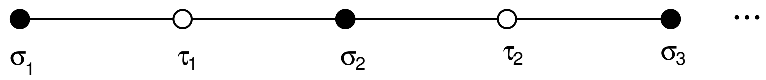
You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. (a) [6 marks] Describe the transfer matrix method for calculating the partition function for a system of spins $\sigma_1, \ldots, \sigma_L$ with $\sigma_j = \pm 1$ on a ring (i.e. we impose periodic boundary conditions) described by an energy of the form $E = \sum_{j=1}^{L} E_0(\sigma_j, \sigma_{j+1})$ with $E_0(\sigma_j, \sigma_{j+1}) = E_0(\sigma_{j+1}, \sigma_j)$.

   (b) [10 marks] Now consider an Ising-like model comprised of alternating spins $\sigma_j = \pm 1$ and $\tau_j = 0, \pm 1$ (the lattice has altogether $2L$ sites and we impose periodic boundary conditions)



   The energy is given by

$$E = -J \sum_{j=1}^{L} \sigma_j \tau_j + \tau_j \sigma_{j+1} \ , \tag{1}$$

   where $J > 0$ and we impose periodic boundary conditions $\sigma_{L+1} = \sigma_1$. Calculate the partition function for model defined in (1) at $T > 0$ by means of the transfer matrix method. What is the free energy per site in the thermodynamic limit?

   (c) [5 marks] Give an explicit calculation of the thermal average of σ_1 in the limit of large L.

   (d) [4 marks] Calculate the thermal average of τ_1^2 in the limit of large L.

2. Consider a one-dimensional quantum spin model with Hamiltonian

$$H = -J\sum_{i=1}^{L} S_i^x S_{i+1}^x + S_i^y S_{i+1}^y - hS\sum_{i=1}^{L} S_i^x \ , \tag{2}$$

where $J, h > 0$ and we have a spin-S on each site of the lattice, i.e. $(S_i^x)^2 + (S_i^y)^2 + (S_i^z)^2 = S(S+1)$. We impose periodic boundary conditions $S_{L+1}^\alpha = S_1^\alpha$.

(a) [5 marks] What is the classical ground state of this model? Justify your answer by a calculation. [Hint: consider (S_j^x, S_j^y, S_j^z) to be a classical vector of length S].

(b) [4 marks] Let $\tilde{S}_j^x$, $\tilde{S}_j^y$, $\tilde{S}_j^z$ be spin-S operators on site j of a one-dimensional lattice. Consider S to be large. The Holstein-Primakoff representation is defined by

$$\tilde{S}_j^z = S - a_j^\dagger a_j, \quad \tilde{S}_j^+ = \tilde{S}_j^x + \imath\tilde{S}_j^y = \sqrt{2S - a_j^\dagger a_j}\ a_j,\ [a_j, a_\ell^\dagger] = \delta_{j,\ell}.$$

Explain the nature and usefulness of this representation. Comment on complications that generally could arise.

(c) [6 marks] Apply the Holstein-Primakoff representation to the Hamiltonian H. How should you choose the spin operators $\tilde{S}_j^\alpha$ to be related to S_j^β and why?
Carry out an expansion of H in inverse powers of S. Ignore the constant contribution and drop all terms that grow more slowly than S, when S becomes large. Show that the resulting Hamitonian $H_{\rm LSW}$, the linear spin wave approximation to H, takes the form

$$H_{\rm LSW} = \sum_{j=1}^{L} A(a_j^\dagger a_{j+1} + a_{j+1}^\dagger a_j) + B(a_j a_{j+1} + a_j^\dagger a_{j+1}^\dagger) + C a_j^\dagger a_j,$$

and determine A, B and C.

(d) [6 marks] Show that $H_{\rm LSW}$ can be written in the form (you may drop constant contributions).

$$H_{\rm LSW} = \sum_k \epsilon(k) b^\dagger(k) b(k) + \text{const.}\ ,$$

where $b^\dagger(k)$, $b(k)$ are bosonic creation and annihilation operators.

(e) [4 marks] Obtain an expression for the expectation values

$$\langle \text{GS}|S_j^\alpha|\text{GS}\rangle\ , \quad \alpha = x, y, z$$

in terms of a momentum sum and discuss the meaning of your result.