A15272W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

ADVANCED QUANTUM FIELD THEORY

Trinity Term 2022

Tuesday, 19th April 2022, 9:30am-12:30am

You should submit answers to all three questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

You are permitted to use the following material(s):
Calculator (candidate to provide)
The use of computer algebra packages is **not** *allowed.*
One summary sheet of A4 notes

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. While the Standard Model Higgs boson cannot couple directly to photons, it can do so indirectly via e.g. an intermediate top quark loop. The topology of the lowest order interaction is shown in Fig. 1.

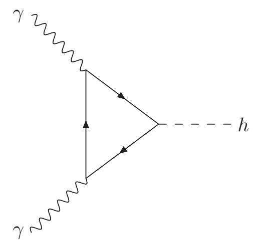

Figure 1: Topology for lowest order $\gamma\gamma h$ interaction in the Standard Model.

Consider the $\gamma(p_1)\gamma(p_2) \to h(p_h)$ amplitude at $O(\lambda_t e^2)$. For this question, only consider the contribution from an intermediate top quark loop.

(i) [2 marks] Draw the contributing Feynman diagrams for this process, with appropriate momentum labelling.

(ii) [3 marks] Show that the scattering amplitude is given by

$$\mathcal{M} = \epsilon^\mu(p_1)\epsilon^\nu(p_2)\left[\mathcal{M}_{\mu\nu}(p_1,p_2) + \mathcal{M}_{\nu\mu}(p_2,p_1)\right] ,$$

with

$$\mathcal{M}_{\mu\nu}(p_1,p_2) \propto \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{\mathrm{Tr}\left[\gamma_\mu(\not{l}+m_t)\gamma_\nu(\not{l}-\not{p}_2+m_t)(\not{p}_1+\not{l}+m_t)\right]}{(l^2-m_t^2)((p_1+l)^2-m_t^2)((l-p_2)^2-m_t^2)} .$$

(iii) [8 marks] Assuming that the photons are on–shell and transversely polarized, show that the scattering amplitude can be written as

$$\mathcal{M} = \epsilon^\mu(p_1)\epsilon^\nu(p_2)\mathcal{M}'_{\mu\nu}(p_1,p_2) ,$$

with

$$\mathcal{M}'_{\mu\nu}(p_1,p_2) \propto \int \frac{\mathrm{d}^4 l}{(2\pi)^4} \frac{A l_\mu l_\nu + B p_{1,\nu}p_{2,\mu} + C(m_t^2 - \frac{m_h^2}{2} - l^2)g_{\mu\nu}}{(l^2-m_t^2)((p_1+l)^2-m_t^2)((l-p_2)^2-m_t^2)} ,$$

and determine the integer constants A, B, C, as well as the overall prefactor.

(iv) [6 marks] Using dimensional regularization in $D = 4-\epsilon$ dimensions, find the relationship between A and C that must be satisfied in order for the above quantity to be finite in the $\epsilon \to 0$ limit, i.e. in 4 dimensions. Verify that this relationship is indeed satisfied, and explain why this must be the case.

(v) [6 marks] Given the above relationship between A and C is satisfied, identify the relationship between B and C that must be satisfied in order for the tensor structure of the amplitude to satisfy

$$\mathcal{M}'_{\mu\nu}(p_1,p_2) \propto \frac{m_h^2}{2}g_{\mu\nu} - p_{1,\nu}p_{2,\mu} ,$$

in the $\epsilon \to 0$ limit, i.e. in 4 dimensions. Verify that B and C do indeed satisfy the required relationship, and comment on your result [*you may leave any Feynman parameter integrals unevaluated*].

[*Selected Feynman rules* :

- *Higgs–top quark vertex:* $-i\frac{\lambda_t}{\sqrt{2}}\delta_{\alpha\beta}\delta_{ij}$, where i, j are the colour indices in the fundamental representation, α, β are the corresponding spinor indices and λ_t is the top quark Yukawa coupling.
- *Top quark propagator:* $i\frac{\not{p}+m_t}{p^2-m_t^2}\delta_{ij}$, where i, j are the colour indices in the fundamental representation.
- *Photon–top quark vertex:* $-iee_t\delta_{ij}\gamma^\mu_{\alpha\beta}$, where i, j are the colour indices in the fundamental representation, e_t is the fractional top quark charge, and α (β) are the spinor indices associated with the fermion line pointing away from (towards) the vertex.

Feynman's formula to combine denominators:

$$\frac{1}{A_1 \cdots A_n} = (n-1)! \int_0^1 \mathrm{d}x_1 \cdots \mathrm{d}x_n \frac{\delta(x_1 + \cdots + x_n - 1)}{(x_1 A_1 + \cdots + x_n A_n)^n} \, . \tag{1}$$

In $D = 4 - \epsilon$ dimensions,

$$\int \frac{\mathrm{d}^D k}{(2\pi)^D} \frac{k^{2a}}{(k^2 + X)^b} = i(-1)^{a-b} \frac{1}{(4\pi)^{D/2}} \frac{1}{(-X)^{b-a-\frac{D}{2}}} \frac{\Gamma\left(a + \frac{D}{2}\right)\Gamma\left(b - a - \frac{D}{2}\right)}{\Gamma(b)\Gamma\left(\frac{D}{2}\right)} \, ,$$

where $\Gamma(\epsilon) = \frac{1}{\epsilon} + O(\epsilon^0)$.

Trace identities:

$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$
$\mathrm{Tr}(\gamma^\mu\gamma^\nu) = 4g^{\mu\nu}$
$\mathrm{Tr}(\text{odd no. of } \gamma^\mu\text{s}) = 0.$]

2. (a) [6 marks] The beta function for a generic coupling α can be written as

$$\beta(\alpha) = \frac{\partial \alpha}{\partial \ln \mu} = b_0 \alpha^2 \,,$$

to $O(\alpha^2)$, where μ is the renomalization scale and b_0 is a constant. Derive an expression for the coupling at scale μ_f in terms of the coupling at a scale μ_i, and discuss the different regimes which exist depending on b_0, providing examples from the gauge couplings of the Standard Model.

(b) (i) [8 marks] Consider the QCD scattering process $u(p_1)\overline{u}(p_2) \to d(p_3)\overline{d}(p_4)$ at tree-level, where u, d are up and down quarks. Draw the corresponding Feynman diagram(s). Neglect quark masses throughout. Averaging over initial–state spins/colours and summing over final–state spins/colours, show that the squared matrix element is proportional to

$$\langle |\mathcal{M}_{u\overline{u}\to d\overline{d}}|^2 \rangle = C \frac{t^2+u^2}{s^2} \,, \tag{2}$$

where s, t, u are Mandelstam variables. Determine the overall constant C.

(ii) [5 marks] Now consider the QED scattering process $e^-(p_1)e^+(p_2) \to e^-(p_3)e^+(p_4)$ at tree–level. Averaging over initial–state spins and summing over final–state spins, the squared matrix element is proportional to

$$\langle |\mathcal{M}_{e^+e^-\to e^+e^-}|^2 \rangle \propto \frac{t^2+u^2}{s^2} + \frac{s^2+u^2}{t^2} + \frac{2u^2}{ts} \,. \tag{3}$$

Draw the contributing Feynman diagrams, and making use of (2), identify the origin of the three terms in the above expression in terms of these. You should justify your answer, but no explicit calculation of $\mathcal{M}_{e^+e^-\to e^+e^-}$ is required.

(iii) [6 marks] Now consider the QCD scattering process $u(p_1)\overline{u}(p_2) \to u(p_3)\overline{u}(p_4)$ at tree–level, again neglecting quark masses throughout. Using the results of (2) and (3), and averaging over initial–state spins/colours and summing over final–state spins/colours, calculate the corresponding squared matrix element [*Hint: you do not have to perform any explicit spinor algebra*].

[*Selected QCD Feynman rules in Feynman–t' Hooft gauge*:

- *Gluon propagator* : $\frac{-ig_{\mu\nu}}{k^2}\delta^{ab}$, where a, b are colour indices in the adjoint representation.
- *Gluon–quark vertex:* $igT^a_{ij}\gamma^\mu_{\alpha\beta}$, where the colour indices i (j) in the fundamental representation are associated with the fermion line pointing away from (towards) the vertex, a is the colour index in the adjoint representation, and α, β are the corresponding spinor indices.

The QCD generators and structure constants satisfy:

- $\mathrm{Tr}(T^aT^b) = \frac{1}{2}\delta^{ab}$.
- $T^a_{ij}T^a_{jk} = \frac{N_C^2-1}{2N_C}\delta_{ik}$, where a and j are summed over.
- $f^{abc}f^{abd} = N_C\delta^{cd}$, where a and b are summed over.]

3. (a) (i) [3 marks] State Goldstone's theorem.

(ii) [7 marks] Consider the following $SO(3)$ symmetric Lagrangian

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi_i\partial^\mu\phi_i - V(\phi_i)$$

with

$$V(\phi) = -\frac{1}{2}\mu^2(\phi_i\phi_i) + \frac{1}{4}\lambda(\phi_i\phi_i)^2 \ ,$$

where ϕ_i are a set of real scalar fields, with $i = 1, 2, 3$ and summation over repeated indices i is implied. The parameters $\mu^2, \lambda > 0$.
Show that the ground state, determined from the minimum of the potential $V(\phi)$, breaks the above symmetry spontaneously. Expanding around a conveniently chosen vacuum–expectation value v, evaluate the masses of the particles in the theory in terms of μ. Comment on your result in light of Goldstone's theorem.

(iii) [8 marks] Now consider the modified potential

$$\tilde{V}(\phi_i) = V(\phi_i) + c\phi_1$$

with parameter $c < 0$, and $V(\phi)$ given as in part (ii). Find the ground state, determined from the minimum of the potential $\tilde{V}(\phi)$, and show that this occurs for a vacuum–expectation value $\tilde{v} > 0$. Expanding around this, evaluate the masses of the particles in the theory in terms of μ, c and $\tilde{v}$. Comment on your result in light of Goldstone's theorem [*Hint: you do not need to solve explicitly for* $\tilde{v}$].

(b) [7 marks] Consider the case of a spontaneously broken abelian gauge symmetry. The terms in the Lagrangian relevant to the massive gauge boson are

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{2}M^2A^\mu A_\mu \ ,$$

in the unitary gauge, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, and M is the gauge boson mass. Using the path integral formalism, compute the gauge boson propagator.