A15272W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

ADVANCED QUANTUM FIELD THEORY FOR PARTICLE PHYSICS

Trinity Term 2021

WEDNESDAY, 21ST APRIL 2021, Opening Time 09:30 am UK Time

You should submit answers to all three questions.

You have **3 hours** *writing time to complete the paper and up to* **30 minutes** *technical time for uploading your file. The allotted technical time must not be used to finish writing the paper.*
Mode of completion (format in which you will complete this exam): **handwritten**
You are permitted to use the following material(s):
Calculator (candidate to provide)
The use of computer algebra packages is **not** *allowed.*

1. Consider scalar QED with no quartic scalar interaction, i.e. $\mathcal{L}_{\text{sQED}} = \mathcal{L}_0' + \mathcal{L}_{ct} + \mathcal{L}_1$, and

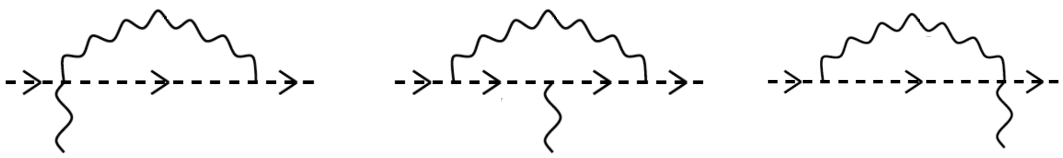
$$\mathcal{L}_0' = \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi - \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \,, \tag{1}$$

$$\mathcal{L}_{ct} = \delta_2 \partial_\mu \phi^* \partial^\mu \phi - \delta_m m^2 \phi^* \phi - \delta_3 \frac{1}{4} F^{\mu\nu} F_{\mu\nu} \,, \tag{2}$$

$$\mathcal{L}_1 = -i Z_1 e \left[\phi^* \partial^\mu \phi - \phi \partial^\mu \phi^* \right] A_\mu + Z_4 e^2 \phi^* \phi A^\mu A_\mu \,, \tag{3}$$

with $\delta_i \equiv Z_i - 1$.

Consider the 1–loop corrections to the $\gamma\phi\phi^*$ vertex, $\mathbf{V}^\mu_{1-\text{loop}}(k, k')$, shown in Fig. 1. We will consider throughout the special case that the incoming scalar has zero momentum, $k' = 0$, for general outgoing scalar momentum, k.



(a) (b) (c)

Figure 1: 1–loop corrections to $\gamma\phi\phi^*$ vertex

We will work in the R_ξ gauge, i.e. the photon propagator is given by

$$-i \frac{g_{\mu\nu} - (1 - \xi) \frac{p_\mu p_\nu}{p^2}}{p^2} \,,$$

for internal photon carrying momentum p. All other Feynman rules are as given in the notes.

(i) [2 marks] For the purposes of calculating the 1–loop corrections to $O(\frac{1}{\epsilon})$, why are we free to make the particular momentum assignment above?

(ii) [7 marks] Consider the contribution to the 1–loop correction to the vertex from Fig. 1 (a). Working always at $O(\frac{1}{\epsilon})$, evaluate the corresponding contribution, $\mathbf{V}^\mu_{1-\text{loop},a}(k, 0)$, to the vertex, and verify that this is independent of ξ.

(iii) [4 marks] Now consider the contribution to the 1–loop correction to the vertex due to Fig. 1 (c). Show that that is zero for arbitrary ξ.

(iv) [9 marks] Consider the contribution to the 1–loop correction to the vertex from Fig. 1 (b), again to $O(\frac{1}{\epsilon})$. Show that this is proportional to

$$\mathbf{V}^\mu_{1-\text{loop},b}(k, 0) \propto \xi \int \frac{d^4 l}{(2\pi)^4} \frac{(2(l \cdot k) + l^2)(2l + k)^\mu}{l^2((l + k)^2 - m^2)(l^2 - m^2)} \,.$$

Show that the second term, $\sim l^2$, in the numerator gives zero contribution, and evaluate the corresponding ξ–dependent correction.

Hence, write down the renormalization constant Z_1 at 1–loop, to $O(\frac{1}{\epsilon})$.

(v) [3 marks] Hence, and without explicit calculation, write down the renormalization constant Z_2, in the same gauge as above, to $O(\frac{1}{\epsilon})$. Do the above results imply that observable quantities are ξ–dependent at 1–loop? Explain your answer.

2. (a) Consider the scattering process $e^-(p_1)e^+(p_2) \to t(p_3)\bar{t}(p_4)$, where t is a top quark.

   (i) [3 marks] Show that
$$s + t + u = 2(m_e^2 + m_t^2) \; ,$$
   where s, t, u are the usual Mandelstam variables.

   (ii) [10 marks] Do not neglect either the electron or top quark masses. Working to leading order in QED, averaging over initial–state spins, polarizations and summing over final–state colours, spins and polarizations, show that the squared matrix element is proportional to
$$\left\langle |\mathcal{M}|^2 \right\rangle \propto A(u^2 + t^2) + Bs(m_e^2 + m_t^2) + C(m_e^2 + m_t^2)^2 \; .$$
   Determine the overall constants A, B, C.

   (iii) [3 marks] Now consider the process $u(p_1)\overline{u}(p_2) \to t(p_3)\bar{t}(p_4)$, where u is an up quark. Working to leading–order in QCD, average over initial–state colours, spins and polarizations, and sum over final–state colours, spins and polarizations. Using the result from part (ii), determine the corresponding squared matrix element.

   (iv) [4 marks] Consider the same $u(p_1)\overline{u}(p_2) \to t(p_3)\bar{t}(p_4)$ but including both QED and QCD interactions, at leading order in both cases. Average over initial–state colours, spins and polarizations, and sum over final–state colours, spins and polarizations. Using the result from part (ii), determine the corresponding squared matrix element.

(b) [5 marks] Consider QED in the so–called ‘t Hooft–Veltman gauge, with the gauge fixing function:
$$f(A) = \partial_\mu A^\mu(x) + \lambda A_\mu(x) A^\mu(x) - \sigma(x) \; , \tag{4}$$
for an arbitrary real parameter λ. This gauge introduces cubic and quartic photon interactions.

Given the Feynman rules for these photon interactions below, draw all diagrams contributing to the photon–photon scattering amplitude for the $\gamma\gamma \to \gamma\gamma$ process at tree–level. Demonstrate that the corresponding scattering amplitude vanishes.

Selected Feynman rules in in ‘t Hooft–Veltman gauge:

- *Photon cubic vertex:* $2\lambda(g^{\beta\gamma}p_1^\alpha + g^{\alpha\gamma}p_2^\beta + g^{\alpha\beta}p_3^\gamma)$, for incoming momenta p_1, p_2, p_3 associated with vertices α, β, γ, respectively.
- *Photon quartic vertex:* $-4i\lambda^2(g^{\alpha\beta}g^{\delta\gamma} + g^{\alpha\gamma}g^{\beta\delta} + g^{\alpha\delta}g^{\beta\gamma})$, where $\alpha, \beta, \gamma, \delta$ label the four vertices.
- *Photon propagator:* $\frac{-ig_{\mu\nu}}{k^2}$.

3. (a) Consider a $SU(2) \times U(1)_Y$ gauge theory with a $Y = 0$, $SU(2)$ triplet of real scalar fields, $(\Phi)_i = \phi_i$, where Y is the $U(1)$ charge operator. The scalar potential is given by

$$V(\Phi) = -\frac{1}{2}m^2\Phi^T\Phi + \lambda(\Phi^T\Phi)^2 ,$$

with $m^2, \lambda > 0$. After SSB, the electrically neutral ($Q = 0$) member of the scalar triplet acquires a vacuum expectation value (where $Q = T^3 + Y$).

(i) [6 marks] Identify the subgroup that remains unbroken, and calculate the Higgs boson mass in this model.

(ii) [7 marks] Calculate the vector boson masses and deduce the Feynman rule for the three–point vertex between the Higgs and the vector bosons.

(iii) [2 marks] Would this be a suitable candidate theory to describe the observed physics of the Standard Model? Explain your answer with a suitable example.

[*The generators of* $SU(2)$ *for the triplet (adjoint) representation are given by* $(T^a)_{bc} = -i\epsilon_{abc}$.]

(b) Consider the theory of a Yukawa interaction between a real scalar field $\phi(x)$ and a spinor field $\psi(x)$. The interaction Lagrangian is given by

$$\mathcal{L}_{\text{int}}(\phi, \overline{\psi}, \psi) = g\phi(x)\overline{\psi}(x)\psi(x) ,$$

where g is a real coupling constant. The generating functional can be written as

$$Z[J, \overline{\eta}, \eta] = \exp\left[i\int \mathrm{d}^4x\, \mathcal{L}_{\text{int}}\left(\frac{1}{i}\frac{\delta}{\delta J(x)}, i\frac{\delta}{\delta \eta(x)}, \frac{1}{i}\frac{\delta}{\delta \overline{\eta}(x)}\right)\right] Z_0[J, \overline{\eta}, \eta] , \qquad (5)$$

where Z_0 is the generating functional for the free theory, given by

$$Z_0[J, \overline{\eta}, \eta] = Z_0[0, 0, 0] \exp\left(\frac{i}{2}\int \mathrm{d}^4x\mathrm{d}^4y J(x)\Delta(x-y)J(y)\right)$$
$$\exp\left(i\int \mathrm{d}^4w\mathrm{d}^4z\overline{\eta}(w)S_F(w-z)\eta(z)\right) ,$$

where $\Delta(x - y)$ and $S_F(w - z)$ are the scalar and fermion propagators, respectively.

(i) [8 marks] Consider in (5) the action of the $O(g^2)$ term in the expansion of $e^{i\int \mathrm{d}^4x\mathcal{L}_{\text{int}}}$ on Z_0. Show that, to $O(g^2)$, the connected scalar two–point correlation function is

$$\langle 0|T\phi(x_a)\phi(y_a)|0\rangle_{O(g^2)} = g^2\int \mathrm{d}^4x\mathrm{d}^4y\Delta(x_a - x)\Delta(y_a - y)\mathrm{Tr}\left[S(x-y)S(y-x)\right] ,$$

where the trace is over the spinor indices.

(ii) [2 marks] The above expression corresponds to the connected single fermion loop correction to the scalar two–point correlation function. How would it change if the fields $\overline{\psi}$, ψ were real scalars? [No explicit calculation is required].