Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

ADVANCED QUANTUM FIELD THEORY FOR PARTICLE PHYSICS

Trinity Term 2020

FRIDAY, 5TH JUNE 2020, 09:30 am

You should submit answers to all three questions.

You have **4 hours** *to complete the paper and upload your answer file.*
You are permitted to use the following material(s):
Calculator
The use of computer algebra packages is **not** *allowed*

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

1. The Lagrangian density for spinor QED has the form

$$\mathcal{L} = i\overline{\psi}\not{D}\psi - m\overline{\psi}\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \ , \tag{1}$$

where $D_\mu = \partial_\mu + ieA_\mu$.

(i) [3 marks] Write down the corresponding renormalized Lagrangian density, in terms of renormalization factors or counterterms.

(ii) [3 marks] Show that

$$\gamma^\nu\gamma^\mu\gamma_\nu = (2-D)\gamma^\mu \ , \tag{2}$$

in D dimensions.

(iii) [1 mark] Draw the Feynman diagram/diagrams contributing to the 1–loop correction to the QED vertex.

(iv) [1 mark] Write down the result for the QED vertex, $iV^\mu(p,p')$, up to $O(e^3)$, where p (p') is the incoming (outgoing) fermion momentum. Do not perform the loop integration at this stage.

(v) [5 marks] Now set the external momenta to zero, $p = p' = 0$, and work in $D = 4 - \epsilon$ dimensions. Show that the 1–loop contribution to the QED vertex is proportional to

$$iV^\mu_{1-\text{loop}}(0,0) \propto \gamma^\mu \int \frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{\alpha k^2 + \beta m^2}{k^2(k^2-m^2)^2} \ , \tag{3}$$

where α, β are ϵ–dependent constants. Determine these and all other proportional factors. [*Do not include an artificial photon mass m_γ here, or in the following calculation*]

(vi) [3 marks] Show that the second term, proportional to β, is finite in $D = 4$ dimensions.

(vii) [4 marks] Calculate the renormalization factor, Z_1, for the QED vertex, to $O(\frac{1}{\epsilon})$.

(viii) [5 marks] Now, consider the 1–loop corrections to the QCD vertex. Draw all contributing Feynman diagrams, and calculate the corresponding colour factors.

[*Selected Feynman rules in Feynman gauge*:

- *Photon propagator :* $\frac{-ig_{\mu\nu}}{k^2}$.
- *Photon–lepton vertex:* $-ie\gamma^\mu$.
- *Lepton propagator:* $i\frac{\not{p}+m}{p^2-m^2}$.

Feynman's formula to combine denominators:

$$\frac{1}{A_1\cdots A_n} = (n-1)!\int_0^1 \mathrm{d}x_1\cdots\mathrm{d}x_n \frac{\delta(x_1+\cdots+x_n-1)}{(x_1A_1+\cdots+x_nA_n)^n} \ . \tag{4}$$

In $D = 4-\epsilon$ dimensions,

$$\int \frac{\mathrm{d}^Dk}{(2\pi)^D}\frac{k^{2a}}{(k^2+X)^b} = i(-1)^{a-b}\frac{1}{(4\pi)^{D/2}}\frac{1}{(-X)^{b-a-\frac{D}{2}}}\frac{\Gamma\left(a+\frac{D}{2}\right)\Gamma\left(b-a-\frac{D}{2}\right)}{\Gamma(b)\Gamma\left(\frac{D}{2}\right)} \ ,$$

where $\Gamma(\epsilon) = \frac{1}{\epsilon} + O(\epsilon^0)$.]

2. (a) Consider the Lagrangian density $\mathcal{L}_{\text{QED}} = \mathcal{L}_0 + \mathcal{L}_I$ for spinor QED, where $\mathcal{L}_0$ is the free Lagrangian density and the interaction Lagrangian density is given by

$$\mathcal{L}_I = -e\overline{\psi}\not{A}\psi \,. \tag{5}$$

(i) [3 marks] Show that the generating functional can be written as

$$Z[J,\eta,\overline{\eta}] = \exp\left[-ie\int \mathrm{d}^4x \left(\frac{1}{i}\frac{\delta}{\delta J^\mu(x)}\right)\left(i\frac{\delta}{\delta\overline{\eta}_\alpha(x)}\right)\gamma^\mu_{\alpha\beta}\left(\frac{1}{i}\frac{\delta}{\delta\eta_\beta(x)}\right)\right] Z_0[J,\eta,\overline{\eta}] \,, \tag{6}$$

where Z_0 is the generating functional for the free theory, μ is a Lorentz index and α,β are spinor indices.

(ii) [7 marks] We have

$$Z_0[J,\eta,\overline{\eta}] = Z_0[0,0,0]\exp\left[i\int \mathrm{d}^4w\mathrm{d}^4z\,\overline{\eta}(w)S(w-z)\eta(z)\right] \cdot \exp\left[\frac{i}{2}\int d^4xd^4yJ_\mu(x)\Delta^{\mu\nu}(x-y)J_\nu(y)\right] \,, \tag{7}$$

where $\Delta^{\mu\nu}(x-y)$ and $S(w-z)$ are the photon and fermion propagators, respectively. Show that, to leading order we have

$$\langle 0|\mathrm{T}\{A_\nu(x)\psi_\alpha(y)\overline{\psi}_\beta(z)\}|0\rangle = -e\int \mathrm{d}^4x_a\Delta_{\mu\nu}(x_a-x)\Big(\mathrm{Tr}\left[\gamma^\mu S(0)\right]S(y-z)_{\alpha\beta} - \left[S(y-x_a)\gamma^\mu S(x_a-z)\right]_{\alpha\beta}\Big) \,, \tag{8}$$

where μ,ν are Lorentz indices, α,β are spinor indices and the trace is over spinor indices. Explain the origin of the two terms on the right hand side.

(iii) [3 marks] Using the explicit expression for the massive fermion propagator, show that the first term, $\propto S(0)$, vanishes.

(b) Let $\mathcal{M}$ be the tree–level scattering amplitude for the scattering process $u(p_1)\gamma(p_2) \to u(p_3)\gamma(p_4)$, where u is an up quark. Take the high energy limit throughout, neglecting quark masses.

(i) [1 mark] Draw all contributing Feynman diagrams.

(ii) [4 marks] Writing

$$\mathcal{M} = \epsilon^\mu(p_2)\epsilon^\nu(p_4)^*\mathcal{M}_{\mu\nu} \,, \tag{9}$$

calculate $p_2^\mu\mathcal{M}_{\mu\nu}$ and comment on the result.

(iii) [7 marks] Averaging over initial–state colours/spins/polarizations and summing over final–state colours/spins/polarizations, show that the squared matrix element is proportional to

$$\langle|\mathcal{M}|^2\rangle = -C\frac{u^2+s^2}{su} \,, \tag{10}$$

where s,u are Mandelstam variables. Determine the overall constant C.

[*Recall that the gamma matrices obey:* $\{\gamma^\mu,\gamma^\nu\} = 2g^{\mu\nu}\cdot\mathbb{1}$, *where* $\mathbb{1}$ *is a unit matrix in spinor space.*
You may also assume without derivation:
$\mathrm{Tr}(\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma) = 4(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma} + g^{\mu\sigma}g^{\nu\rho})$, $\gamma^\mu\not{a}\gamma_\mu = -2\not{a}$, $\gamma^\mu\not{a}\not{b}\gamma_\mu = 4(a\cdot b)$, and $\gamma^\mu\not{a}\not{b}\not{c}\gamma_\mu = -2\not{c}\not{b}\not{a}$.
Massless quark propagator: $i\frac{\not{p}}{p^2}\delta_{ij}$.
Photon–quark vertex: $-iee_q\delta_{ij}\gamma^\mu$, where i,j are the colour indices in the fundamental representation and e_q is the fractional quark charge.]

3. (a) (i) [2 marks] State Goldstone's theorem.

(ii) [8 marks] Consider a spontaneously broken $SU(2)$ gauge theory with a complex scalar doublet ϕ. The scalar potential is given by

$$V(\phi) = \lambda \left(\phi^\dagger \phi - \frac{1}{2} v^2 \right)^2 . \tag{11}$$

Show that the gauge symmetry is spontaneously broken, and suitably expanding around the scalar field vacuum expectation value

$$\langle \phi(x) \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} , \tag{12}$$

determine the masses of all particles in the broken theory as well as all new interactions introduced by this breaking. Comment on your result in light of Goldstone's theorem. [*The covariant derivative of the scalar field is given by* $(D_\mu \phi)_i = \partial_\mu \phi_i - igA^a_\mu \tau^a_{ij} \phi_j$, where $\tau^a = \frac{1}{2}\sigma^a$ and σ^a are the Pauli spin matrices.]

(b) Consider QED in the so–called 't Hooft–Veltmann gauge. That is, consider the gauge fixing function:

$$f(A) = \partial_\mu A^\mu(x) + \lambda A_\mu(x) A^\mu(x) - \sigma(x) , \tag{13}$$

for an arbitrary real parameter λ.

(i) [3 marks] Explain the origin of the Faddeev–Popov determinant. Considering the action of an infinitesimal gauge transformation, show that for the gauge fixing function (13) this cannot be dropped from the path integral for $\lambda \neq 0$.

(ii) [3 marks] Explain the role of ghosts in calculating the Faddeev–Popov determinant, and determine the corresponding ghost Lagrangian density in the 't Hooft–Veltmann gauge.

(iii) [2 marks] By performing a path integral over σ, weighted by the function $\exp[-\frac{i}{2\xi} \int \mathrm{d}^4 x \sigma^2(x)]$, determine the corresponding gauge fixing Lagrangian term.

(iv) [2 marks] By identifying the corresponding terms in the Lagrangian density, verify that this gauge will introduce cubic and quartic photon interactions.

(v) [5 marks] Draw all Feynman diagrams contributing to the photon–photon scattering amplitude for the $\gamma\gamma \to \gamma\gamma$ process at tree–level. Demonstrate that the corresponding scattering amplitude vanishes. You may take $\xi = 1$, with the corresponding Feynman rules given below.

Selected Feynman rules in the 't Hooft–Veltmann gauge with $\xi = 1$:

- *Photon cubic vertex:* $2\lambda(g^{\beta\gamma} p_1^\alpha + g^{\alpha\gamma} p_2^\beta + g^{\alpha\beta} p_3^\gamma)$, for incoming momenta p_1, p_2, p_3 associated with vertices α, β, γ, respectively.
- *Photon quartic vertex:* $-4i\lambda^2(g^{\alpha\beta} g^{\delta\gamma} + g^{\alpha\gamma} g^{\beta\delta} + g^{\alpha\delta} g^{\beta\gamma})$, where $\alpha, \beta, \gamma, \delta$ label the four vertices.
- *Photon propagator:* $\frac{-ig_{\mu\nu}}{k^2}$.