A15272W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

ADVANCED QUANTUM FIELD THEORY FOR PARTICLE PHYSICS

Trinity Term 2017

WEDNESDAY, 19TH APRIL 2017, 2.30pm to 5.30pm

You should submit answers to all three questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. The Lagrangian for scalar electrodynamics has the form

$$L = -(D^\mu\phi)^\dagger (D_\mu\phi) - m^2\phi^\dagger\phi - \tfrac{1}{4}\lambda\left(\phi^\dagger\phi\right)^2 - \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

where $D_\mu = \partial_\mu - ieA_\mu$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ and the metric is $g_{\mu\nu} = Diagonal(-1,+1,+1,+1)$

(a) [3 marks] What are the counter-terms needed to renormalise the theory?

(b) [2 marks] Draw the Feynman graphs generating the one-loop corrections to the scalar propagator.

(c) [4 marks] Show that in $d = 4-\varepsilon$ dimensions, in the $\lambda = 0$ limit, the one loop corrections to the propagator have the form

$$i\Pi_\phi\left(k^2\right) = 4e^2\mu^\varepsilon \int \frac{d^dl}{(2\pi)^d} \frac{P_{\alpha\beta}(l)\,k^\alpha k^\beta}{l^2\left((l+k)^2+m^2\right)} - 2(d-1)e^2\mu^\varepsilon \int \frac{d^dl}{(2\pi)^d}\frac{1}{l^2+m_\gamma^2}.$$

Here μ is a constant with scaling dimension 1, $P_{\mu\nu}(l) = g_{\mu\nu} - l_\mu l_\nu/l^2$ and m_γ is an infra-red regulator.

(d) [2 marks] Show that the second term on the right hand side (RHS) vanishes in the limit $m_\gamma \to 0$ (with ϵ fixed).

(e) [7 marks] Show that the first term on the RHS can be written as

$$4e^2\mu^\varepsilon \int_0^1 \frac{x}{2}dx \int \frac{d^dq}{(2\pi)^d}\frac{q^2k^2-(q.k)^2}{(q^2+D)^3}$$

where $D = x(1-x)k^2 + xm^2$.

(f) [7 marks] To $O\left(\frac{1}{\varepsilon}\right)$ determine the one-loop counter terms needed to renormalise $\Pi_\phi\left(k^2\right)$.

[*Feynman rules in the Lorentz gauge:*

- *For each internal scalar,* $\frac{-i}{(k^2+m^2)}$.
- *For each internal photon,* $\frac{-iP_{\mu\nu}(k)}{k^2}$.
- *For each scalar-scalar-photon vertex,* $ie(k+k')_\mu$, *where* k *and* k' *are the incoming and outgoing scalar 4-momenta.*
- *For each scalar-scalar-photon-photon vertex,* $-2ie^2g_{\mu\nu}$.

Feynman's formula to combine denominators:

$$\frac{1}{A\,B} = \int_0^1 dx \frac{1}{(xA+(1-x)B)^2}$$

In $d = 4-\epsilon$ *dimensions*

$$\int \frac{d^dk}{(2\pi)^d}\frac{k^{2a}}{(k^2+\Delta)^b} = i\frac{1}{(4\pi)^{d/2}}\frac{1}{\Delta^{b-a-\frac{d}{2}}}\frac{\Gamma\left(a+\frac{d}{2}\right)\Gamma\left(b-a-\frac{d}{2}\right)}{\Gamma(b)\,\Gamma\left(\frac{d}{2}\right)}$$

where $\Gamma(\varepsilon) = \frac{1}{\varepsilon} + O(\epsilon^0)$.]

2. (a) Consider a field theory of a real pseudoscalar field $\phi(x)$ coupled to the electron field $\psi(x)$. The interaction Lagrangian is

$$L_{\text{int}} = -i\lambda\overline{\psi}\,(x)\,\gamma_5\psi\,(x)\,\phi\,(x)$$

where λ is a real coupling constant.

(i) [3 marks] Show that the generating functional can be written as

$$Z[J,\overline{\varsigma},\varsigma] = \int D\phi D\overline{\psi}D\psi \exp\left(iL_{\text{int}}\left(\frac{1}{i}\frac{\delta}{\delta J(x)}, i\frac{\delta}{\delta\varsigma(x)}, \frac{1}{i}\frac{\delta}{\delta\overline{\varsigma}(x)}\right)\right) Z_0[J,\overline{\varsigma},\varsigma]$$

given that

$$Z_0[J,\overline{\varsigma},\varsigma] = \exp\left(-\tfrac{i}{2}\int d^4x d^4y J(x)\Delta(x-y)J(y)\right).\exp\left(-i\int d^4z d^4w\,\overline{\varsigma}(z)S_F(z-w)\varsigma(w)\right)$$

is the generating functional for the free theory and $\Delta_F(y-x)$ and $S_F(y-x)$ are the scalar and fermion propagators respectively.

(ii) [6 marks] Show that, to leading order,

$$\begin{aligned}\langle 0|\,T\left[\phi(y)\psi_\rho(z)\overline{\psi}_\sigma(\omega)\right]|0\rangle = \lambda(\gamma_5)_{\alpha\beta}\textstyle\int d^4x i\Delta_F\,(y-x)\,[iS_F(x-\omega)_{\beta\sigma}iS_F(z-x)_{\rho\alpha}\\ -iS_F(0)_{\beta\alpha}iS_F(z-\omega)_{\rho\sigma}]\end{aligned}$$

Why are there two terms in the time ordered product?

(b) Define T to be the scattering amplitude for the process $q(k_1)\overline{q}(k_2) \to t(p_1)\overline{t}(p_2)$ (where $q \neq t$ is any light quark and t is the top quark).

(i) [8 marks] At $O(\alpha_s^2)$ in the strong coupling, α_s, show that

$$|T|^2 \propto Tr[\gamma_\mu(\not{p}_2 - M)\gamma_\alpha(\not{p}_1 - M)].Tr[\gamma^\mu \not{k}_1\gamma^\alpha \not{k}_2]$$

where $Tr[..]$ denotes the trace. Determine the constant of proportionality.
In your calculation, average over initial colours and spins and sum over final colours and spins. You may assume the initial quark and anti-quark are massless, but do not neglect the mass, M, of the top quark.

(ii) [8 marks] Evaluate the traces and determine $|T|^2$ in terms of the centre-of-mass energy $\sqrt{s}$ and the squared four momentum t, where $s = -(k_1+k_2)^2$ and $t = -(k_1-p_1)^2$.

[

$$Tr\left(T^aT^b\right) = \tfrac{1}{2}\delta_{ab}$$

$$T^a_{ij}T^a_{kl} = \tfrac{1}{2}\left(\delta_{il}\delta_{jk} - \tfrac{1}{3}\delta_{ij}\delta_{kl}\right)$$

In the R_ξ gauge with metric $g_{\mu\nu} = Diagonal(-1,+1,+1,+1)$ the gluon propagator has the form

$$\widetilde{\Delta}^{ab}_{\mu\nu}\,(k) = -i\frac{\delta^{ab}}{k^2 - i\varepsilon}\left(g_{\mu\nu} - \frac{k_\mu k_\nu}{k^2} + \xi\frac{k_\mu k_\nu}{k^2}\right)$$

and the gluon quark vertex is given by $ig\gamma^\mu T^a_{ij}$.]

3. (a) The Lagrangian describing a spontaneously broken abelian gauge theory has the form

$$L = -(D^\mu \phi)^\dagger D_\mu \phi - V(\phi) - \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu}$$

where $D_\mu = \partial_\mu - igA_\mu$, the potential is given by

$$V(\phi) = \tfrac{1}{4}\lambda \left(\phi^\dagger \phi - \tfrac{1}{2} v^2\right)^2,$$

and ϕ is a complex scalar field that, after spontaneous symmetry breaking, has the form $\phi = \frac{1}{\sqrt{2}}(v + h + ib)$ where h and b are real scalar fields.

The R_ξ gauge is defined by adding the gauge fixing and ghost terms of the form

$$L_{gf} + L_{gh} = -\tfrac{1}{2}\xi^{-1} G^2 - \bar{c}\frac{\partial G}{\partial \theta} c$$

where

$$G = \partial^\mu A_\mu - \xi g v b$$

and θ parameterises an infinitesimal gauge transformation,

$$\begin{aligned} A_\mu &\to A_\mu - \partial_\mu \theta, \\ \phi &\to \phi - ig\theta\, \phi. \end{aligned}$$

(i) [5 marks] Explain the origin of the ghost term.

(ii) [5 marks] Determine the mass and coupling of the ghost field.

(b) Consider a spontaneously broken $SU(2) \times U(1)$ gauge theory with a $Y = 0$, $SU(2)$ triplet of real scalar fields, $(\Phi)_i = \phi_i$, where Y is the $U(1)$ charge operator. The scalar potential is given by

$$V(\Phi) = -\tfrac{1}{2} m^2 \Phi^T \Phi + \lambda \left(\Phi^T \Phi\right)^2.$$

(i) [5 marks] Show that when Φ acquires a vacuum expectation value with $\langle \phi_i \rangle = v_i$ and the symmetry is spontaneously broken, if $\sum_{k=1}^{3} (T^a)_{jk} v_k \neq 0$, then $\sum_{k=1}^{3} (T^a)_{jk} v_k$ is an eigenvector of the mass squared matrix, $\left.\frac{\partial^2 V(\Phi)}{\partial \phi_i \partial \phi_j}\right|_{\Phi = \langle \Phi \rangle}$ with eigenvalue zero.

(ii) [10 marks] Assuming the electrically neutral ($Q = 0$) member of the scalar triplet acquires a vacuum expectation value (where $Q = T^3 + Y/2$), compute the vector boson masses and identify the subgroup that remains unbroken. Determine the identity and mass of the physical Higgs scalar in this model.

[*The generators of* $SU(2)$ *for the triplet representation are given by* $(T^a)_{bc} = -i\epsilon_{abc}$]