A15272W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

ADVANCED QUANTUM FIELD THEORY FOR PARTICLE PHYSICS

Trinity Term 2016

TUESDAY, 19 APRIL 2016, 14.30 to 17.30

You should submit answers to all three questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. (a) [10 marks] The free Lagrangian in spinor electrodynamics has the form

$$L_0 = i\bar{\psi}\partial\!\!\!/\psi - m\bar{\psi}\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}$$

(i) Add the terms necessary to obtain the gauge-invariant Lagrangian density, including counter terms.

(ii) Show that the one-loop contribution to the inverse fermion propagator is proportional to

$$e^2 \int \frac{d^4 l}{(2\pi)^4} \left[\gamma^\nu \tilde{S}(p\!\!\!/ + l\!\!\!/)\gamma^\mu \tilde{\Delta}_{\mu\nu}(l)\right] \tag{1}$$

where e is the fermion charge and $\tilde{S}$ and $\tilde{\Delta}_{\mu\nu}$ are the momentum space fermion and photon propagators respectively.

(iii) Using dimensional regularisation in $d = (4 - \epsilon)$ dimensions and the Feynman gauge, determine the wave function and mass counter terms to $O(\frac{1}{\epsilon})$.

[*In Feynman gauge the photon propagator has the form* $\Delta_{\mu\nu}(l) = \frac{-ig_{\mu\nu}}{(l^2 - i\epsilon)}$ *and the fermion propagator is given by* $\tilde{S}(p\!\!\!/) = \frac{-i(-p\!\!\!/ + m)}{(p^2 + m^2 - i\epsilon)}$. *In* $d = (4 - \epsilon)$ *dimensions* $\gamma_\mu\gamma^\mu = -d, \gamma_\mu p\!\!\!/\gamma^\mu = (d-2)p\!\!\!/$ *and* $\int \frac{d^d q}{(2\pi)^d} \frac{1}{(q^2 + D)^2} = \frac{i}{8\pi^2}\frac{1}{\epsilon}$ *+finite as* $\epsilon \to 0$.]

(b) [10 marks] Consider the quark-antiquark pair production in QCD. Specifically, focus on the $u\bar{u} \to d\bar{d}$ process so there is only one tree-level Feynman diagram contributing to this process.

(i) Draw this diagram. Summing over final state spins and averaging over initial state spins calculate the modulus-squared scattering amplitude, $|\mathcal{M}|^2$, and show that it is proportional to $\frac{t^2 + u^2}{s^2}$, where s, t and u are the Mandelstam variables.

(ii) Sum/average the $|\mathcal{M}|^2$ over the *colours* of the final/initial particles and determine the full form of $|\mathcal{M}|^2$.

[*For simplicity, neglect the quark masses.*]

2. The fully gauge fixed generating functional for QED is given by

$$Z = \langle 0|0\rangle = \int DA^\mu\, D\psi\, D\bar{\psi}\, D\eta\, D\omega\;\; \exp(i\, S_{QED}) = \int DA^\mu\, D\psi\, D\bar{\psi}\, D\eta\, D\omega\; \times$$
$$\exp\left(i \int d^4x[-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\zeta}\,(\partial_\mu A^\mu)^2 + \eta\partial^\mu\partial_\mu\omega + \bar{\psi}(i\gamma^\mu D_\mu - m)\psi]\right),$$

where $D_\mu = \partial_\mu - i\,e\,A_\mu$ is the covariant derivative, and e is the electron charge.

(a) [7 marks] Show that it is invariant under the BRST transformations

$$\begin{aligned}
&A^\mu(x) \to A'^\mu(x) = A^\mu(x) + \epsilon\,\partial^\mu\omega(x)\,,\\
&\bar{\psi}(x) \to \bar{\psi}'(x) = \bar{\psi}(x) - i\,e\,\epsilon\,\omega(x)\,\bar{\psi}(x)\,,\\
&\psi(x) \to \psi'(x) = \psi(x) + i\,e\,\epsilon\,\omega(x)\,\psi(x)\,,\\
&\eta(x) \to \eta'(x) = \eta(x) + \frac{1}{\zeta}\,\partial_\mu A^\mu(x)\,\epsilon \equiv \eta(x) + \delta_\epsilon\eta\,,\\
&\omega(x) \to \omega'(x) = \omega(x)\,,
\end{aligned}$$

where A_μ is the gauge field, ψ and $\bar{\psi}$ are the fermions fields, and η and ω are the ghost fields. In this context ϵ is a Grassmann parameter. [*Hint: To show the Lagrangian is invariant it helps to use the manifest gauge invariance of part of the Lagrangian. Note that it is also necessary to check the invariance of the path integral measure.*]

(b) [4 marks] By computing the variation under the BRST symmetry of a general time ordered product of fields $H(A^\mu, \bar{\psi}, \psi, \eta, \omega)$ derive the Ward identity

$$\langle 0|T\left(\frac{\partial H}{\partial A^\mu}\epsilon\partial^\mu\omega(x) + \frac{\partial H}{\partial\eta}\delta_\epsilon\eta - ie\frac{\partial H}{\partial\bar{\psi}}\epsilon\omega\bar{\psi} + ie\frac{\partial H}{\partial\psi}\epsilon\omega\psi\right)|0\rangle = 0.$$

(c) [9 marks] Explain why the ghosts play no role in QED.

Determine the form of the ghost Lagrangian for the case of a nonabelian gauge theory with the choice of the gauge fixing function, $G^a(x) = (\partial^\mu A^a_\mu(x) - \omega^a(x))$, appropriate for the R_ξ gauge.

Finally by performing a path integral over ω, weighted by the function $\exp\left[-\frac{i}{2\xi}\int d^4x\;\omega^a\omega^a\right]$, show how one obtains the R_ξ gauge and determine the gauge fixing Lagrangian term.

3. (a) [7 marks] Consider a theory with a nonabelian gauge symmetry and also a U(1) gauge symmetry. The theory contains left-handed Weyl fields in the representations (R_i, Q_i), where R_i is the representation of the nonabelian group, and Q_i is the $U(1)$ charge. By considering the group theory factors, find the conditions for this theory to be anomaly free. [*Note that it not necessary to compute the full Feynman diagrams.*]

In the Standard Model there are three families of left-handed Weyl fields transforming as $(1, 2, -\frac{1}{2}) \oplus (1, 1, +1) \oplus (3, 2, \frac{1}{6}) \oplus (\bar{3}, 1, -\frac{2}{3}) \oplus (\bar{3}, 1, \frac{1}{3})$ under the Standard Model gauge group. Show that the Standard Model is anomaly free. [*You must consider 3-3-3, 2-2-2, 3-3-1, 2-2-1 and 1-1-1 anomalies, where the number denotes the gauge group of one of the external gauge fields in the triangle diagram. Why do we not need to worry about the unlisted combinations?*]

(b) [7 marks] The Standard Model contains a complex scalar field, ϕ, the Brout, Englert, Higgs (BEH) boson, in the representation $(2, -\frac{1}{2})$ of the electroweak part of the gauge group, $SU(2) \times U(1)$. The BEH potential has the form

$$V(\phi) = \frac{1}{4}\lambda(\phi^\dagger \phi - \frac{1}{2}v^2)^2.$$

Starting with the scalar field vacuum expectation value in the form

$$\phi(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} v \\ 0 \end{pmatrix},$$

show that the gauge symmetry is spontaneously broken and determine the eigenstates of the resulting massive gauge bosons in terms of the weak mixing angle, θ_W, where $\theta_W = \tan^{-1}(g_1/g_2)$, and g_2, g_1 are the gauge couplings of the $SU(2)$ and $U(1)$ gauge groups respectively. Show that the ratio of the masses of the charged to neutral weak interaction gauge bosons is given by $\cos(\theta_W)$ and determine the electromagnetic coupling. [*The covariant derivative of the Higgs field has the form*

$$(D_\mu \phi)_i = \partial_\mu \phi_i - i[g_2 A^a_\mu T^a + g_1 B_\mu Y]_i^j \phi_j,$$

where $T^a = \frac{1}{2}\sigma^a$, $Y = -\frac{1}{2}I$ and σ^a and I are the Pauli spin matrices and the unit matrix respectively.]

(c) [6 marks] The Lagrangian for a massive Abelian gauge boson has the form

$$L_0 = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{2}M^2 A^\mu A_\mu$$

Using the path integral formalism, compute the gauge boson propagator in the unitary gauge. What is the drawback of the unitary gauge?