A15271W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

ADVANCED FLUID DYNAMICS

Trinity Term 2022

Tuesday, 19th April 2022, 9:30am-11:30am

You should submit answers to both of the two questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

You are permitted to use the following material(s):
Calculator (candidate to provide)
The use of computer algebra packages is **not** *allowed.*
One summary sheet of A4 notes

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. **Magnetohydrodynamics.**

This is a model which was developed in the 1960s to calculate the angular momentum carried away by the solar wind as a result of the torque exerted by the solar magnetic field on the wind flow (L. Mestel, 1961, MNRAS, 122, 473).

We use cylindrical coordinates (r, θ, z) and assume axial symmetry and steady–state. We write the magnetic field and velocity as $\mathbf{B} = \mathbf{B}_p + \mathbf{B}_\theta$ and $\mathbf{v} = \mathbf{v}_p + \mathbf{v}_\theta$, where the subscripts '$p$' and '$\theta$' indicate the poloidal (in the (r, z)–plane) and toroidal components, respectively. We define the angular velocity Ω such that $v_\theta = r\Omega$.

For a scalar ψ and vector $\mathbf{u}$, you may assume, in cylindrical coordinates:

$$\boldsymbol{\nabla}\psi = \left(\frac{\partial \psi}{\partial r}, \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \frac{\partial \psi}{\partial z} \right), \quad \boldsymbol{\nabla}\cdot\mathbf{u} = \frac{1}{r}\frac{\partial (r u_r)}{\partial r} + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z},$$

$$\boldsymbol{\nabla}\times\mathbf{u} = \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}, \frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}, \frac{1}{r}\frac{\partial (r u_\theta)}{\partial r} - \frac{1}{r}\frac{\partial u_r}{\partial \theta} \right).$$

(a) [7 marks] Justify that $\boldsymbol{\nabla}\times(\mathbf{v}\times\mathbf{B}) = \mathbf{0}$. Using this relation, show that $\boldsymbol{\nabla}\times(\mathbf{v}_p\times\mathbf{B}_p) = \mathbf{0}$ and $\boldsymbol{\nabla}\times(\mathbf{v}_p\times\mathbf{B}_\theta + \mathbf{v}_\theta\times\mathbf{B}_p) = \mathbf{0}$. Show that the first equation implies $\mathbf{v}_p = \kappa\mathbf{B}_p$, where κ is a scalar (*hint*: if the curl of a vector is zero, the vector can be written as the gradient of a scalar). Show that the second equation implies that

$$\mathbf{B}\cdot\boldsymbol{\nabla}\left(\Omega - \frac{\kappa B_\theta}{r}\right) = 0,$$

that is to say the quantity in parentheses, which we note Ω_0, is constant along field lines. Comment on this result when $\kappa = 0$ (this is called Ferraro's law).

(b) [3 marks] Using the mass conservation equation, show that $\rho\kappa \equiv \eta$ is constant along field lines.

(c) [7 marks] Write the θ–component of the equation of motion, and show that

$$r^2\Omega - \frac{rB_\theta}{\eta\mu_0} \equiv l$$

is constant along field lines. Interpret the terms in this expression.

(d) [5 marks] Give an expression for Ω along a field line as a function of r, ρ and the constants defined above. Discuss the cases $B_p^2/\mu_0 \gg \rho v_p^2$ and $B_p^2/\mu_0 \ll \rho v_p^2$ (explain the physical meaning of these terms).

(e) [3 marks] Using the expression for Ω found above, calculate l at the Alfvén critical point r_A, defined as the radius where $v_p = v_{A,p}$, with $v_{A,p}$ being the poloidal component of the Alfvén velocity. Express l as a function of r_A and Ω_0.

2. **Complex Fluids.**

This question is about Stokes flow in an incompressible fluid with dynamic viscosity μ.

(a) [4 marks] Suppose that the fluid occupies a volume V. The fluid velocity $\mathbf{u} = \mathsf{E} \cdot \mathbf{x}$ on the boundary ∂V, where E is a symmetric, traceless tensor that depends only upon time. Show that

$$\int_V \mathsf{e}\, \mathrm{d}V = |V|\, \mathsf{E},$$

where the strain rate e has components $e_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i)$, and $|V|$ is the volume of V.

(b) [4 marks] Now suppose that the volume V encloses a particle occupying the volume V_p. The particle is made of a linear elastic material that deforms uniformly, so that $\mathbf{u} = \dot{\mathsf{D}} \cdot \mathbf{x}$ in V_p, where $\dot{\mathsf{D}}$ is the derivative of a symmetric, traceless tensor D that depends only upon time. Show that

$$\int_{V_\mathrm{f}} \mathsf{e}\, \mathrm{d}V = |V|\, \mathsf{E} - |V_\mathrm{p}|\, \dot{\mathsf{D}},$$

where $V_\mathrm{f} = V \setminus V_\mathrm{p}$ is the volume outside the particle occupied by fluid

(c) [4 marks] The stress inside the volume V_p is $\boldsymbol{\sigma} = -p\mathsf{I} + 2G\,\mathsf{D}$, where p is the pressure, I is the identity, and G is a constant. Show that the average stress in the volume V is

$$\langle \boldsymbol{\sigma} \rangle = -\langle p \rangle \mathsf{I} + 2\mu\, \mathsf{E} + \phi\, (2G\, \mathsf{D} - 2\mu\, \dot{\mathsf{D}}),$$

where $\phi = |V_\mathrm{p}|/|V|$ is the fraction of the volume V occupied by the particle. The average of a quantity $\cdots$ over the volume V is

$$\langle \cdots \rangle = \frac{1}{|V|} \int_V \cdots \,\mathrm{d}V.$$

(d) [9 marks] Now suppose that the volume V_p comprises a small sphere in the centre of a cube V. The deformation of the location of the boundary ∂V_p is negligibly small, so any necessary boundary conditions can be imposed on the surface of the undeformed sphere. By considering a flow of the form $\mathbf{u} = \dot{\mathsf{D}} \cdot \mathbf{x} + \widetilde{\mathbf{u}}$ in V_f, show that the normal stress in the fluid just outside ∂V_p is

$$\boldsymbol{\sigma} \cdot \mathbf{n} = -p\,\mathbf{n} + \mu\, \mathsf{E} \cdot \mathbf{n} - 3\mu\, \dot{\mathsf{D}} \cdot \mathbf{n}.$$

Use continuity of the normal stress to show that D evolves according to

$$\mathsf{D} + \tau\, \dot{\mathsf{D}} = \frac{5}{3}\tau\, \mathsf{E}, \text{ where } \tau = \frac{3\mu}{2G}.$$

(e) [4 marks] Show that the volume average of the traceless part of the stress, $\mathsf{S} = \left\langle \boldsymbol{\sigma} - \frac{1}{3}\mathsf{I}\,\mathrm{Tr}\,\boldsymbol{\sigma} \right\rangle$, evolves according to

$$\mathsf{S} + \tau\, \dot{\mathsf{S}} = 2\mu \left(1 + \frac{5}{2}\phi\right) \mathsf{E} + 2\mu \left(1 - \frac{5}{3}\phi\right) \tau\, \dot{\mathsf{E}},$$

and comment on the behaviour as $\tau \to 0$.

[*Hint: The Stokes flow* $\widetilde{\mathbf{u}}$ *around a rigid sphere of radius* a *that satisfies the boundary conditions*

$$\widetilde{\mathbf{u}} = \mathbf{0} \text{ on } |\mathbf{x}| = a, \quad \widetilde{\mathbf{u}} \sim \widetilde{\mathsf{E}} \cdot \mathbf{x} \text{ as } |\mathbf{x}| \to \infty$$

produces a stress $\widetilde{\boldsymbol{\sigma}}$ *with normal component* $\widetilde{\boldsymbol{\sigma}} \cdot \mathbf{n} = 5\mu\, \widetilde{\mathsf{E}} \cdot \mathbf{n}$ *on* $|\mathbf{x}| = a$.]