A15271W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

ADVANCED FLUID DYNAMICS

Trinity Term 2021

TUESDAY, 20TH APRIL 2021, Opening Time 09:30 am UK Time

You should submit answers to both questions.

You have **2 hours** *writing time to complete the paper and up to* **30 minutes** *technical time for uploading your file. The allotted technical time must not be used to finish writing the paper.*
Mode of completion (format in which you will complete this exam): **handwritten**
You are permitted to use the following material(s):
Calculator (candidate to provide)
The use of computer algebra packages is **not** *allowed.*

1. **Magnetohydrodynamics.** In certain models of magnetised plasmas, collisions are not sufficiently strong to isotropise pressure with respect to the local direction $\mathbf{b} = \mathbf{B}/|\mathbf{B}|$ of the magnetic field $\mathbf{B}$. It is assumed, therefore, that pressure is a diagonal matrix

$$\mathsf{p} = p_\perp(\mathsf{I} - \mathbf{bb}) + p_\parallel \mathbf{bb} \tag{1}$$

[in index notation, this means $p_{ij} = p_\perp(\delta_{ij} - b_i b_j) + p_\parallel b_i b_j$]. The perpendicular ($p_\perp$) and parallel ($p_\parallel$) pressures are then calculated either kinetically or from some appropriate (or at least plausible) fluid model.

(a) [10 marks] Consider a static equilibrium with constant, uniform magnetic field $\mathbf{B}_0 = B_0\hat{\mathbf{z}}$, density ρ_0, and pressures $p_{\perp 0} = p_{\parallel 0} = p_0$ (isotropic equilibrium pressure). Start from the MHD equations for density, magnetic field and fluid velocity, the latter with the anisotropic pressure force $-\nabla\cdot\mathsf{p}$ instead of the usual gradient of isotropic pressure. Adopt the Reduced-MHD ordering, in which relative perturbations of all fields are the same order as $k_\parallel/k_\perp \ll 1$. Show that the reduced equations for the Alfvénic fields (perturbations of the velocity, $\mathbf{u}_\perp$, and magnetic field, $\delta\mathbf{B}_\perp$, perpendicular to $\mathbf{B}_0$) are unaffected by the introduction of the anisotropic pressure, while the perturbations of the magnetic-field strength (δB), density ($\delta\rho$), parallel velocity ($u_\parallel$), and parallel pressure ($\delta p_\parallel$) are related by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\delta B}{B_0} - \frac{\delta\rho}{\rho_0}\right) = \mathbf{b}\cdot\nabla u_\parallel, \qquad \rho_0\frac{\mathrm{d}u_\parallel}{\mathrm{d}t} = -\mathbf{b}\cdot\nabla\delta p_\parallel, \tag{2}$$

where $\mathrm{d}/\mathrm{d}t = \partial/\partial t + \mathbf{u}_\perp\cdot\nabla$ and $\mathbf{b}\cdot\nabla = \partial/\partial z + (\delta\mathbf{B}_\perp/B_0)\cdot\nabla_\perp$.

[For this open-book exam, you need not rederive, and may use, any results derived in the Lecture Notes, if you state them clearly.]

(b) [10 marks] A model often employed to calculate the anisotropic pressure in such contexts is the so-called double-adiabatic, or Chew–Goldberger–Low (CGL), equations,

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{p_\perp}{\rho B} = 0, \qquad \frac{\mathrm{d}}{\mathrm{d}t}\frac{p_\parallel B^2}{\rho^3} = 0, \tag{3}$$

which express the particles conserving certain adiabatic invariants of motion. Use the linearised version of these equations, and that of equations (2), to find the dispersion relation for the slow waves in the CGL approximation and to show that, in the limit of $\beta = 8\pi p_0/B_0^2 \gg 1$, these "slow waves" in fact propagate much faster than Alfvén waves.

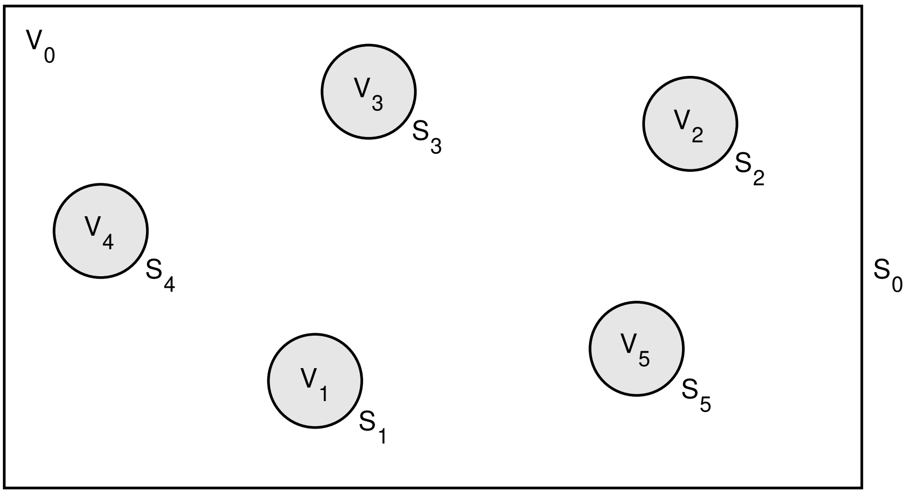
(c) [5 marks] Explain in what way the physics of these CGL slow waves is different from the physics of the slow waves in standard MHD with isotropic pressure, and why, therefore, they are able to propagate faster.

2. **Complex Fluids.** This is a question about Stokes flow for an incompressible fluid with viscosity μ and no body forces. The dissipation due to viscosity in a volume V of fluid is

$$\Phi = 2\mu \int_V e_{ij} e_{ij} \, \mathrm{d}V, \text{ where } e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right).$$

Consider a volume V_0 of fluid with a prescribed velocity on the outer boundary S_0. The resulting Stokes flow in V_0 has velocity $\mathbf{u}^{(0)}$, stress $\boldsymbol{\sigma}^{(0)}$, and dissipation $\Phi^{(0)}$.

Now suppose that rigid particles are introduced into the flow, to occupy volumes $V_1, \ldots, V_N$ with boundaries $S_1, \ldots, S_N$. The velocity prescribed on the outer boundary S_0 is unchanged. The resulting Stokes flow in the remaining volume $V_{\text{fluid}} = V_0 \setminus (V_1 \cup \cdots \cup V_N)$ has velocity $\mathbf{u}$, stress $\boldsymbol{\sigma}$, and dissipation Φ.



(a) [3 marks] Show that the dissipation in any volume V of fluid with boundary S is

$$\Phi = \int_S u_i \sigma_{ij} n_j \, \mathrm{dS},$$

where $\mathbf{n}$ is the unit normal pointing out of the fluid on S. Give a physical interpretation of this result.

(b) [3 marks] Show that the viscous dissipation in the flow around the particles is

$$\Phi = \int_{S_{\text{all}}} \mathbf{u}^{(0)} \cdot \boldsymbol{\sigma} \cdot \mathbf{n} \, \mathrm{dS} + \sum_{p=1}^{N} \int_{S_p} \left(\mathbf{u} - \mathbf{u}^{(0)}\right) \cdot \boldsymbol{\sigma} \cdot \mathbf{n} \, \mathrm{dS},$$

where $S_{\text{all}} = S_0 \cup S_1 \cup \cdots \cup S_N$ is the boundary of V_{fluid}.

(c) [9 marks] Show further that $\Phi = \Phi^{(0)} + \Phi'$, where

$$\Phi' = \sum_{p=1}^{N} \int_{S_p} \mathbf{u} \cdot \boldsymbol{\sigma}^{(0)} \cdot \mathbf{n} + \left(\mathbf{u} - \mathbf{u}^{(0)}\right) \cdot \boldsymbol{\sigma} \cdot \mathbf{n} \, \mathrm{dS},$$

and show that the contribution from $\mathbf{u} \cdot \boldsymbol{\sigma}^{(0)} \cdot \mathbf{n}$ vanishes for rigid particles.

[*You may use the reciprocal theorem without proof provided you state it clearly.*]

Question continues overleaf…

(d) [4 marks] Now suppose that the particles are small compared with the size of V_0, so we can expand the velocity field $\mathbf{u}_0$ in the region V_p occupied by particle p with centre $\mathbf{X}_p$ as

$$\mathbf{u}^{(0)}(\mathbf{x}) = \mathbf{u}_p^{(0)} + \boldsymbol{\Omega}_p^{(0)} \times (\mathbf{x} - \mathbf{X}_p) + \mathsf{E}_p^{(0)} \cdot (\mathbf{x} - \mathbf{X}_p) + \cdots ,$$

where $\boldsymbol{\Omega}_p^{(0)}$ is a vector and $\mathsf{E}_p^{(0)}$ is a symmetric traceless matrix. Hence show that

$$\Phi' = \sum_{p=1}^{N} \mathsf{E}_p^{(0)} : \int_{S_p} (\mathbf{x} - \mathbf{X}_p)\, \boldsymbol{\sigma} \cdot \mathbf{n} \ \mathrm{dS},$$

for particles with no external forces or torques acting upon them. The : denotes a double contraction between two rank-2 tensors.

(e) [6 marks] Finally, suppose that the particles are widely separated small spheres of radius a, and that the boundary conditions on S_0 are $\mathbf{u}^{(0)} = \mathsf{E}^{(0)} \cdot \mathbf{x}$, where $\mathsf{E}^{(0)}$ is a symmetric traceless matrix.

You may assume that the normal stress on the surface of each sphere is

$$\boldsymbol{\sigma} \cdot \mathbf{n} = 5\mu\, \mathsf{E}^{(0)} \cdot \mathbf{n}.$$

Evaluate Φ' for this system, and hence derive the Einstein effective viscosity $\mu_E = \mu \left(1 + \frac{5}{2}\phi\right)$ for a dilute suspension of rigid spheres with volume fraction ϕ.