Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

ADVANCED FLUID DYNAMICS

Trinity Term 2020

MONDAY, 1ST JUNE 2020, 09:30 am

You should submit answers to both of the two questions.

You have **3 hours** *to complete the paper and upload your answer file.*
You are permitted to use the following material(s):
Calculator
Formula Sheet
The use of computer algebra packages is **not** *allowed*

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

1. **Magnetohydrodynamics.**

   (a) [5 marks] State the equations of ideal, incompressible magnetohydrodynamics ("iMHD") and explain under what physical and mathematical assumptions they are valid.

   (b) [8 marks] Consider the restriction of these equations to two dimensions (2D), i.e., assume that the velocity $\mathbf{u} = (u_x, u_y, 0)$ and the magnetic field $\mathbf{B} = (B_x, B_y, 0)$ both lie in the plane (x, y) and do not depend on the third coordinate z. Show that the 2D magnetic field can be described by the vector potential $\mathbf{A} = (0, 0, A)$ and that the quantity $\int_S \mathrm{d}x\mathrm{d}y\, A^2$ is conserved (except for resistive dissipation). The integral is over some area S and you may assume that there are no flows through its boundary. This 2D invariant is sometimes called "anastrophy".

   (c) [6 marks] Assume that a 2D iMHD system will relax to a static equilibrium with minimum energy subject to constant anastrophy. Show that the equilibrium magnetic field satisfies the equation

$$\nabla^2 A = \alpha A, \tag{1}$$

   where α is a constant. You may assume that all surface terms that need to vanish, do vanish.

   (d) [6 marks] Show that the magnetic field satisfying (1) is force-free, i.e., that it produces no force in the iMHD momentum equation. Thus, you have derived a 2D version of J. B. Taylor relaxation to linear force-free state.

2. **Complex Fluids.**

This question is about Stokes flow in an incompressible fluid with dynamic viscosity μ.

(a) [8 marks] Show that the velocity field $\mathbf{u} = \nabla\phi \times \mathbf{c}$, with $\mathbf{c}$ a constant vector, satisfies the Stokes equations for a flow with constant pressure when the scalar function ϕ satisfies $\nabla^2\phi = 0$.

Hence, or otherwise, show that the Stokes flow around a sphere of radius a rotating with angular velocity Ω about a vertical axis is given by

$$v_\phi = \frac{a^3}{r^2}\Omega \sin\theta,$$

in standard spherical polar coordinates where θ is the angle from the upward vertical.

Show that the torque on the sphere has magnitude $T = 8\pi\mu a^3\Omega$.

[*Hint: for a purely azimuthal flow,* $\sigma_{r\phi} = \mu\, r\frac{\partial}{\partial r}\left(\frac{v_\phi}{r}\right)$.]

(b) [10 marks] Consider a rigid spheroidal body of mass m whose axis is aligned with the unit vector $\mathbf{p}$. The body is moving with translational velocity $\mathbf{U}$ and angular velocity $\boldsymbol{\Omega}$ relative to a background Stokes flow with velocity $\mathbf{U}^\infty$, strain rate E^∞ and angular velocity $\boldsymbol{\Omega}^\infty$ far from the body. The force $\mathbf{F}$, torque $\mathbf{T}$, and stresslet S exerted by the fluid on the body are given by

$$\begin{pmatrix} \mathbf{F} \\ \mathbf{T} \\ \mathsf{S} \end{pmatrix} = \mu \begin{pmatrix} \mathsf{A} & 0 & \widetilde{\mathsf{G}} \\ 0 & \mathsf{C} & \widetilde{\mathsf{H}} \\ \mathsf{G} & \mathsf{H} & \mathsf{M} \end{pmatrix} \begin{pmatrix} \mathbf{U}^\infty - \mathbf{U} \\ \boldsymbol{\Omega}^\infty - \boldsymbol{\Omega} \\ \mathsf{E}^\infty \end{pmatrix},$$

where

$$C_{ij} = X^C p_i p_j + Y^C(\delta_{ij} - p_i p_j), \quad H_{ijk} = \widetilde{H}_{kij} = \frac{1}{2}Y^H(\epsilon_{ikl}p_j + \epsilon_{jkl}p_i)p_l,$$

and X^C, Y^C, Y^H are three constants determined by the shape of the body. The torque is calculated about the geometrical centre of the body.

The centre of mass of the body is separated by a a vector $-h\mathbf{p}$ from its geometrical centre, so that the body experiences a gravitational torque about its geometrical centre.

Show that the orientation of the body evolves according to

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = \Gamma(\mathbf{k} - \mathbf{k}\cdot\mathbf{p}\,\mathbf{p}) + \boldsymbol{\Omega}^\infty \times \mathbf{p} + \beta\left(\mathsf{E}^\infty\cdot\mathbf{p} - \mathbf{p}\cdot\mathsf{E}^\infty\cdot\mathbf{p}\,\mathbf{p}\right),$$

where $\beta = Y^H/Y^C$ and $\mathbf{k}$ is a unit vector pointing vertically upwards. Find an expression for the constant Γ, and calculate Γ explicitly for a sphere of radius a and mean density $\overline{\rho}$.

(c) [7 marks] Suppose that the unit vector $\mathbf{p}$ is confined to lie in the yz plane with y horizontal and z vertically upwards. The fluid velocity far from the body is $\mathbf{u} = u(z)\hat{\mathbf{y}}$, and the body is small compared with the lengthscale over which u varies. Show that the angle Θ between $\mathbf{p}$ and the y-axis evolves according to

$$\frac{\mathrm{d}\Theta}{\mathrm{d}t} = \Gamma\cos\Theta + \Omega\left(1 - \beta\cos 2\Theta\right),$$

where Ω is half the local vorticity of the fluid flow.

Discuss the possibility of steady solutions for the cases $\beta = 0$ and $\beta = 1$. Which shape particles do these correspond to?