Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

ADVANCED FLUID DYNAMICS

Trinity Term 2018

TUESDAY, 17TH APRIL 2018, 2:30pm to 4:30pm

You should submit answers to both of the two questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. **Magnetohydrodynamics.** Most of this question is not on MHD, but deals with a system of equations describing a somewhat analogous system embedded into an external field and supporting anisotropic wave-like perturbations: incompressible fluid rotating at angular velocity $\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$, where $\hat{\mathbf{z}}$ is the unit vector in the direction of the z axis. The velocity field $\mathbf{u}$ in such a fluid satisfies the following equation

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\nabla p + 2\mathbf{u}\times\boldsymbol{\Omega}, \tag{1}$$

where pressure p is found from the incompressibility condition $\nabla\cdot\mathbf{u} = 0$, the last term on the right-hand side is the Coriolis force, the centrifugal force has been absorbed into p, and viscosity has been ignored.

(a) [7 marks] Consider infinitesimal perturbations of a static ($\mathbf{u}_0 = 0$), homogeneous equilibrium of (1), $\mathbf{u} \propto e^{-i\omega t + i\mathbf{k}\cdot\mathbf{r}}$, where $\mathbf{k} = (k_\perp, 0, k_\parallel)$ (without loss of generality; the subscripts refer to directions perpendicular and parallel to the axis of rotation). Show that the system supports waves with the dispersion relation

$$\omega = \pm 2\Omega\frac{k_\parallel}{k}, \tag{2}$$

where $k = |\mathbf{k}|$. These are called *inertial waves.*

(b) [4 marks] In the case $k_\parallel \ll k_\perp$, determine the direction of propagation of the inertial waves. Determine also the relationship between the components of the velocity vector $\mathbf{u}$ associated with the wave. Comment on the polarisation of the wave.

(c) [9 marks] When rotation is strong, i.e., when $\Omega \gg ku$, perturbations in a rotating system are anisotropic with $\epsilon = k_\parallel/k_\perp \ll 1$. Order the linear and nonlinear time scales to be similar to each other and work out the ordering of all relevant quantities, namely, $\mathbf{u}_\perp$ (horizontal velocity), $u_\parallel$ (vertical velocity), δp (perturbed pressure), ω, Ω, $k_\parallel$, $k_\perp$ with respect to each other and to ϵ. Using this ordering, show that the motions of a rotating fluid satisfy the following reduced equations

$$\frac{\partial}{\partial t}\nabla_\perp^2\Phi + \left\{\Phi, \nabla_\perp^2\Phi\right\} = 2\Omega\frac{\partial u_\parallel}{\partial z}, \tag{3}$$

$$\frac{\partial u_\parallel}{\partial t} + \left\{\Phi, u_\parallel\right\} = -2\Omega\frac{\partial \Phi}{\partial z}, \tag{4}$$

where $\{f, g\} = (\partial_x f)(\partial_y g) - (\partial_y f)(\partial_x g)$ and Φ is the stream function of the perpendicular velocity, i.e., to the lowest order in ϵ, $\mathbf{u}_\perp^{(0)} = \hat{\mathbf{z}}\times\nabla_\perp\Phi$. Note that, in order to obtain the above equations, you will need to work out $\nabla_\perp\cdot\mathbf{u}_\perp$ to both the lowest and next order in ϵ, i.e., both $\nabla_\perp\cdot\mathbf{u}_\perp^{(0)}$ and $\nabla_\perp\cdot\mathbf{u}_\perp^{(1)}$.

(d) [2 marks] Show that any purely horizontal flows in a strongly rotating fluid must be exactly two-dimensional (i.e., constant along the axis of rotation).

(e) [3 marks] For a strongly rotating, incompressible, highly electrically conducting fluid embedded in a strong uniform magnetic field parallel to the axis of rotation, discuss without calculation under what conditions you would expect anisotropic ($k_\parallel \ll k_\perp$) Alfvénic and slow-wave-like (pseudo-Alfvénic) perturbations to be decoupled from each other? The Alfvén speed is v_A.

2. **Complex Fluids.**

(a) [5 marks] Consider two Stokes flows $\mathbf{u}_1$, $\boldsymbol{\sigma}_1$ and $\mathbf{u}_2$, $\boldsymbol{\sigma}_2$ generated by body forces $\mathbf{f}_1$ and $\mathbf{f}_2$ in a volume V with boundary ∂V.

Show that

$$\int_V \mathbf{f}^{(1)} \cdot \mathbf{u}^{(2)} \, \mathrm{d}V + \int_{\partial V} \mathbf{u}^{(2)} \cdot \boldsymbol{\sigma}^{(1)} \cdot \mathbf{n} \, \mathrm{d}S = \int_V \mathbf{f}^{(2)} \cdot \mathbf{u}^{(1)} \, \mathrm{d}V + \int_{\partial V} \mathbf{u}^{(1)} \cdot \boldsymbol{\sigma}^{(2)} \cdot \mathbf{n} \, \mathrm{d}S.$$

(b) [6 marks] Show that the flow

$$p(\mathbf{x}) = \frac{3\mu a}{2} \frac{\mathbf{U} \cdot \mathbf{x}}{r^3}, \quad \mathbf{u}(\mathbf{x}) = \mathbf{U} \left(\frac{3a}{4r} + \frac{a^3}{4r^3} \right) + (\mathbf{U} \cdot \mathbf{x}) \, \mathbf{x} \, \frac{3}{4} \left(\frac{a}{r^3} - \frac{a^3}{r^5} \right),$$

with $r = |\mathbf{x}|$ satisfies the boundary condition(s) required for the flow of a fluid with viscosity μ around a sphere of radius a translating with velocity $\mathbf{U}$ whose centre is instantaneously located at $\mathbf{x} = 0$ in unbounded fluid. Show further that the surface traction $\boldsymbol{\sigma} \cdot \mathbf{n}$ is constant on $r = a$, and hence derive Stokes' formula for the drag force on the sphere.

(c) [10 marks] Consider a Stokes flow $\mathbf{u}_\infty$ generated by body forces in an unbounded fluid. A sphere of radius a is inserted at $\mathbf{x} = 0$. The body forces all lie outside the sphere, and are undisturbed by the insertion of the sphere. Call the resulting Stokes flow $\mathbf{u}_S$.

By applying the reciprocal theorem to the flow in part (b) and to the disturbance $\mathbf{u}_S - \mathbf{u}_\infty$ show that the sphere translates with velocity

$$\mathbf{V} = \frac{1}{4\pi a^2} \int_{|\mathbf{x}|=a} \mathbf{u}_\infty(\mathbf{x}) \, \mathrm{d}S.$$

(d) [4 marks] Show further that

$$\mathbf{V} = \left(1 + \frac{a^2}{6} \nabla^2 \right) \mathbf{u}_\infty \Big|_{\mathbf{x}=0}.$$

[*Hint: consider a Taylor expansion of* $\mathbf{u}_\infty$ *and consider* $\nabla^2 p$.]