A15271W1

Honour School of Mathematical and Theoretical Physics Part C
Master of Science in Mathematical and Theoretical Physics

ADVANCED FLUID DYNAMICS

Trinity Term 2017

TUESDAY, 18TH APRIL 2017, 2:30pm to 4:30pm

You should submit answers to both questions.

You must start a new booklet for each question which you attempt. Indicate on the front sheet the numbers of the questions attempted. A booklet with the front sheet completed must be handed in even if no question has been attempted.

The numbers in the margin indicate the weight that the Examiners anticipate assigning to each part of the question.

Do not turn this page until you are told that you may do so

1. **Magnetohydrodynamics.** The magnetic field in the solar atmosphere can be thought of as consisting of a family of arc-shaped field lines with footpoints rooted on the surface of the Sun. Below, we will ignore the curvature of the solar surface and denote by z the spatial coordinate perpendicular to it (the vertical direction).

   (a) [4 marks] Consider a magnetic field of the type described above and argue that a field configuration with a given set of footpoints can be represented as

   $$\mathbf{B} = \nabla\alpha \times \nabla\beta, \tag{1}$$

   where $\alpha(x, y, z)$ and $\beta(x, y, z)$ are some functions whose values at $z = 0$ are given.

   (b) [9 marks] Show that the field that minimises the magnetic energy within a domain subject to the constraint that α and β are fixed at the boundary of the domain is a force-free field.

   (c) [3 marks] Show that if this field is a linear force-free field, it satisfies the Helmholtz equation

   $$\nabla^2\mathbf{B} + \lambda^2\mathbf{B} = 0, \tag{2}$$

   where λ is a constant.

   (d) [9 marks] Consider the semi-infinite domain $z \geqslant 0$. Find a linear force-free field that decays exponentially with z, is independent of the horizontal coordinate y, has $B_z = B_0$ at $x = z = 0$, and $B_x = B_y = 0$ at $x = 0$ and $x = a$. Sketch this field in the (x, z) and (x, y) planes. For what values of λ does such a field exist?

2. **Complex Fluids.**

(a) [5 marks] Consider a suspension of pairs of spheres joined by Hookean springs with spring constant H. The function $\psi(\mathbf{x}, \mathbf{R}, t)$ specifies a distribution of sphere-spring pairs with separation $\mathbf{R}$, and is normalised so that

$$\int \psi(\mathbf{x}, \mathbf{R}, t)\, \mathrm{d}\mathbf{R} = 1.$$

There are n such sphere-spring pairs per unit volume in three-dimensional space.

Show that the number of springs per unit area with separation $\mathbf{R}$ that intersect a plane with normal $\mathbf{n}$ is

$$n(\mathbf{R} \cdot \mathbf{n})\psi,$$

and hence that the stress due to the stretching of the springs is

$$nH \int \mathbf{R}\mathbf{R}\, \psi(\mathbf{x}, \mathbf{R}, t)\, \mathrm{d}\mathbf{R} \equiv nH\langle \mathbf{R}\mathbf{R} \rangle.$$

(b) [13 marks] The total stress due to the sphere-spring pairs is

$$\boldsymbol{\sigma} = -nk_B T \mathsf{I} + nH\langle \mathbf{R}\mathbf{R} \rangle \qquad (\dagger)$$

at temperature T. The effect of the other sphere-spring pairs in the suspension can be modelled by supposing that a sphere moving with velocity $\mathbf{v}$ relative to its surrounding fluid experiences an anisotropic hydrodynamic resistance force $-\zeta(\mathsf{I} + \alpha\boldsymbol{\sigma})^{-1}\mathbf{v}$.

By formulating equations of motion for the two spheres in a sphere-spring pair, subject to Brownian forces, show that ψ obeys the equation

$$\partial_t \psi + \mathbf{u} \cdot \nabla\psi + \nabla_{\mathbf{R}} \cdot \left(\mathbf{R} \cdot (\nabla \mathbf{u})\psi - \frac{2}{\zeta}(\mathsf{I} + \alpha\boldsymbol{\sigma})\, H\mathbf{R}\psi \right) = \frac{2k_B T}{\zeta} \nabla_{\mathbf{R}} \cdot \left((\mathsf{I} + \alpha\boldsymbol{\sigma}) \cdot \nabla_{\mathbf{R}}\psi \right),$$

where $[\nabla \mathbf{u}]_{ij} = \partial_i u_j$, and hence that $\boldsymbol{\sigma}$ obeys the equation

$$\lambda \left(\partial_t \boldsymbol{\sigma} + \mathbf{u} \cdot \nabla\boldsymbol{\sigma} - \boldsymbol{\sigma} \cdot (\nabla \mathbf{u}) - (\nabla \mathbf{u})^{\mathsf{T}} \cdot \boldsymbol{\sigma} \right) + \boldsymbol{\sigma} + \alpha\boldsymbol{\sigma}^2 = \mu \left((\nabla \mathbf{u}) + (\nabla \mathbf{u})^{\mathsf{T}} \right). \qquad (\star)$$

Give expressions for the coefficients λ and μ, and an interpretation of the expression $(\partial_t \boldsymbol{\sigma} + \cdots)$ on the left-hand side.

What property of this equation for $\boldsymbol{\sigma}$ justifies the coefficient of the isotropic contribution to the stress in $(\dagger)$?

[You may assume that the Brownian force on the sphere at $\mathbf{r}_i$ is $-k_B T\, \nabla_{\mathbf{r}_i} \log\psi$, and neglect any consequent Brownian diffusion in the $\mathbf{x}$ coordinates.]

(c) [7 marks] Suppose the velocity field is $\mathbf{u} = (\gamma x, -\frac{1}{2}\gamma y, -\frac{1}{2}\gamma z)$. Show that there are steady, spatially uniform solutions of $(\star)$ with

$$\boldsymbol{\sigma} = \begin{pmatrix} \sigma_{11} & 0 & 0 \\ 0 & \sigma_{22} & 0 \\ 0 & 0 & \sigma_{22} \end{pmatrix},$$

and find the equations satisfied by σ_{11} and σ_{22}. Comment on the behaviour of σ_{11} as γ increases, for both $\alpha > 0$ and $\alpha = 0$.